Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 61 (2017) 601–606
www.elsevier.com/locate/endm

Sticky matroids and Kantor's Conjecture

Winfried Hochstättler, Michael Wilhelmi [1]

Department of Mathematics and Computer Science
FernUniversität in Hagen
Hagen, Germany

Abstract

We prove the equivalence of Kantor's Conjecture and the Sticky Matroid Conjecture due to Poljak und Turzík.

Keywords: matroid, amalgam, embedding, projective space

1 Introduction

The purpose of this paper is to prove the equivalence of two classical conjectures from combinatorial geometry. Kantor's Conjecture [4] adresses the problem whether a combinatorial geometry can be embedded into a modular geometry, i.e. a direct product of projective spaces. He conjectured that this is impossible only if there exists a non-modular pair of hyperplanes.

The other conjecture, the Sticky Matroid Conjecture (SMC) due to Poljak und Turzík [6] concerns the question whether it is possible to glue two matroids together along a common part. They conjecture that a "common part" for which this is always possible, a sticky matroid, must be modular. It is well-known (see eg. [5]) that modular matroids are sticky and easy to see [6] that

[1] Email: michael.wilhelmi@fernuni-hagen.de

http://dx.doi.org/10.1016/j.endm.2017.07.013

modularity is necessary for ranks up to three. Bachem and Kern [1] proved that a rank-4 matroid which has two hyperplanes intersecting in a point is not sticky. They also stated that a matroid is not sticky if for each of its non-modular pairs there exists an extension decreasing its modular defect. The proof of this statement had a flaw which was fixed by Bonin [2]. Using a result of Wille [7] and Kantor [4] this implies that the sticky matroid conjecture is true if and only if it holds in the rank-4 case. Bonin [2] also showed that a matroid of rank $\mathrm{r} \geq 3$ with two disjoint hyperplanes is not sticky and that non-stickyness is also implied by the existence of a hyperplane and a line which do not intersect but can be made modular in an extension.

We generalize Bonin's result and show that a matroid is not sticky, if it has a non-modular pair which admits an extension decreasing its modular defect. Moreover by showing the existence of the proper amalgam we prove that in the rank-4 case this condition is also necessary for a matroid not to be sticky. As a consequence from every counterexample to Kantor's conjecture arises a matroid, which can be extended in finite steps to a counterexample of the (SMC) implying the eqivalence of the two conjectures. A further consequence of our results is the equivalence of both conjectures to the following:

Conjecture 1.1 *In every finite non-modular matroid there exists a non-modular pair and a point-extension decreasing its modular defect.*

Finally, we present an example proving that the (SMC), like Kantor's Conjecture, fails in the infinite case.

We assume familiarity with matroid theory, the standard reference is [5].

2 Our results

Let M be a matroid with groundset E and rank function r. We define the *modular defect* $\delta(X,Y)$ of a pair of subsets $X,Y \subseteq E$ as

$$\delta(X,Y) = \mathrm{r}(X) + \mathrm{r}(Y) - \mathrm{r}(X \cup Y) - \mathrm{r}(X \cap Y).$$

By submodularity of the rank function the modular defect is always non-negative. If it equals zero, we call (X,Y) a *modular pair*. A matroid is called *modular*, if all pairs of flats form a modular pair.

An *extension* of a matroid M on a set E is a matroid N on a set $F \supseteq E$ such that $M = N|E$. If N_1, N_2 are extensions of a common matroid M with groundsets F_1, F_2 resp. E such that $F_1 \cap F_2 = E$, then a matroid $A(N_1,N_2)$ with groundset $F_1 \cup F_2$ is called an *amalgam* of N_1 and N_2 along