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Anagram-free colorings of graphs

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Abstract

A sequence S is called anagram-free if it contains no segment $r_1 r_2 \dots r_k r_{k+1} \dots r_{2k}$ such that $r_{k+1} \dots r_{2k}$ is a permutation of the block $r_1 r_2 \dots r_k$. Answering a question of Erdős and Brown, Keränen constructed an infinite anagram-free sequence on four symbols. Motivated by the work of Alon, Grytczuk, Hałuszczak and Riordan [1], we consider a natural generalisation of anagram-free sequences for graph colorings. A coloring of the vertices of a given graph G is called *anagram-free* if the sequence of colors on any path in G is anagram-free. We call the minimal number of colors needed for such a coloring the *anagram-chromatic* number of G.

We have studied the anagram-chromatic number of several classes of graphs like trees, minor-free graphs and bounded-degree graphs. Surprisingly, we show that there are bounded-degree graphs (such as random regular graphs) in which anagrams cannot be avoided unless we basically give each vertex a separate color.

Keywords: Random regular graph, coloring, anagram

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1 Introduction

The study of non-repetitive colorings was conceived by a famous result of Thue [6] from 1906. He showed that there exists an infinite sequence S on an alphabet of three symbols in which no two adjacent blocks (of any length) are the same. In other words, S contains no sequence of *consecutive* symbols $r_1 r_2 \dots r_{2n}$ with $r_i = r_{i+n}$ for all $i \leq n$. Note that it is not a priori obvious that the minimal size of the alphabet necessary for an infinite non-repetitive sequence is even finite. Thue's result is interesting in its own right, but it also has influential and surprising applications, the most famous one probably occurring in a solution to the Burnside problem for groups by Novikov and Adjan. Thue-type problems lead to the development of Combinatorics on Words, a new area of research with many interesting connections and applications.

A sequence $r_1 r_2 \dots r_n r_{n+1} \dots r_{2n}$ is called an *anagram* if the second block, $r_{n+1} \dots r_{2n}$, is a permutation of $r_1 r_2 \dots r_n$. A long standing open question of Erdős and Brown was whether there exists a sequence on $\{0, 1, 2, 3\}$ containing no anagrams. After a series of improvements, Keränen [4] constructed arbitrarily long anagram-free sequences on four symbols using Thue's idea – given a non-repetitive finite sequence S on symbols $\{0, 1, 2, 3\}$, we can replace each symbol by a longer word on the same alphabet in a way that yields a new, longer non-repetitive sequence $\overline{S}$. This answered the open question, but at the same time opened new avenues for further studies.

Alon, Grytczuk, Hałuszczak and Riordan [1] proposed another variation on the non-repetitive theme. Let G be a graph. A vertex coloring $c : V(G) \to \mathcal{C}$ is called *non-repetitive* if any path in G induces a non-repetitive sequence. Define the *Thue number* $\pi(G)$ as the minimal number of colors in a non-repetitive coloring of G. It is easy to see that this number is a strengthening of the classical chromatic number, as well as the star-chromatic number. It turns out that the Thue number is bounded for several interesting classes of graphs, e.g. $\pi(P_n) \leq 3$ for a path P_n of length n (directly from Thue's Theorem), and $\pi(T) \leq 4$ for any tree T. Alon *et al.* [1] showed that $\pi(G) \leq c\Delta(G)^2$, where c is a constant and $\Delta(G)$ denotes the maximum degree of G. They also found a graph G with $\pi(G) \geq \frac{c'\Delta^2}{\log \Delta}$. Closing the above gap remains an intriguing open question. Another interesting problem is to decide if the Thue number of planar graphs is finite. A survey of Grytczuk [2] lays out some progress in this direction, as well as numerous related questions on non-repetitive graph

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