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Evolution of the giant component in graphs on orientable surfaces

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Abstract

For a fixed integer $g \geq 0$, let $S_g(n, m)$ be a graph chosen uniformly at random from all graphs with n vertices and m edges that are embeddable on the orientable surface $\mathbb{S}_g$ of genus g. We prove that the component structure of $S_g(n, m)$ features two phase transitions. The first one is analogous to the emergence of the giant component in the classical Erdős-Rényi random graph $G(n, m)$ at $m \sim \frac{n}{2}$. The second phase transition occurs at $m \sim n$, when the giant component covers almost all vertices.

Keywords: Random graphs, surface, phase transition, giant component

1 Introduction

1.1 Motivation

Since Erdős and Rényi first introduced random graphs in their groundbreaking paper [2], asymptotic properties of random graphs have been among the most

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studied phenomena in combinatorics. One of the most prominent examples is the appearance of the *giant component*. In a graph $G(n, m)$ chosen uniformly at random from all graphs with vertex set $[n] := \{1, \dots, n\}$ and m edges, a unique largest component emerges when $m - \frac{n}{2} \gg n^{2/3}$.

Similar phenomena have been observed for various classes of random combinatorial structures. In this regard, a particularly interesting behaviour can be observed for graphs that can be embedded on a given 2-dimensional surface. Random graphs on 2-dimensional surfaces have attained considerable attention since the pioneering work of McDiarmid, Steger, and Welsh [8]. Subsequently, Giménez and Noy [4] and Chapuy et al. [1] proved—among many other results—that random graphs with n vertices and $m = \mu n$ edges embeddable on a given surface have a component that covers all but finitely many vertices, provided that $\mu \in (1, 3)$. Kang and Łuczak [6] considered random planar graphs in the "sparse" regime $\mu \leq 1$ and showed that (i) at $\mu \sim \frac{1}{2}$, a unique largest component emerges and (ii) at $\mu \sim 1$, the number of vertices *outside* the largest component decreases drastically.

The result of [6] immediately raises the question if graphs embeddable on any fixed surface feature analogous phenomena. In addition, [6] covers the regime $m \sim n$ only for $m - n \ll n^{2/3}$, which leaves a gap to the "dense" regime considered in [1,4]. The purpose of this paper is to answer this question and to close the gap to the dense regime.

1.2 Main results

In this paper we show that the largest component in a graph embeddable on the orientable surface $\mathbb{S}_g$ of genus g undergoes two critical phases (analogous to that of planar graphs from [6]). For the rest of the paper, let $g \geq 0$ be an integer and denote by $S_g(n, m)$ a graph chosen uniformly at random from all graphs on the vertex set $[n]$ with m edges that are embeddable on $\mathbb{S}_g$.

We use the standard notation of *with high probability*, or whp for short, whenever an event holds with probability tending to 1 as n tends to infinity. A connected graph is called *complex* if it contains at least two cycles. Our first main result states that the giant component in $S_g(n, m)$ emerges at the same time as for the Erdős-Rényi random graph $G(n, m)$.

Theorem 1.1 *Suppose that*

$$m = m(n) = \frac{n}{2}\left(1 + \lambda n^{-1/3}\right), \quad \text{where } \lambda = \lambda(n) \to \infty, \text{ but } \lambda \ll n^{1/3}.$$

Then the largest component H_1 of $S_g(n, m)$ whp is complex for any $g \geq 0$ and it is not embeddable on $\mathbb{S}_{g-1}$ if $g \geq 1$. Furthermore, for every function