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Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 61 (2017) 703–709 www.elsevier.com/locate/endm

Lattice walks in the octant with infinite associated groups

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Abstract

Continuing earlier investigations of restricted lattice walks in $\mathbb{N}^3$, we take a closer look at the models with infinite associated groups. We find that up to isomorphism, only 12 different infinite groups appear, and we establish a connection between the group of a model and the model being Hadamard.

Keywords: Lattice Walk, Infinite Group, Regular Expression.

1 Introduction

Since the classification project for nearest neighbor lattice walk models in the quarter plane, initiated by Bousquet-Melou and Mishna [5], is largely completed, the analogous question for 3D models in the octand is getting into the focus [1,2,7]. Given a stepset $\mathcal{S} \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$, let $f(x,y,z,t) = \sum_{n,i,j,k} a_{i,j,k,n} x^i y^j z^k t^n$ be the generating function which counts the number $a_{i,j,k,n}$ of walks in $\mathbb{N}^3$ from $(0,0,0)$ to (i,j,k) consisting of n steps taken from $\mathcal{S}$. The main question is then: for which choices $\mathcal{S}$ is the series f D-finite?

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http://dx.doi.org/10.1016/j.endm.2017.07.026

For models in 2D, it turns out that the generating function is D-finite if and only if a certain group associated to the model is finite, see [3, 4, 6, 8–12] and the references therein. The situation in 3D seems to be more complicated, as evidenced by some models having a finite group that seem to be non-D-finite [1, 2]. Among the 2^{3^3-1} models, there are (up to bijection) 10,908,263 models which have a group associated to them. For 10,905,833 of these models, their group has more than 400 elements. It was shown in [7] for all the models with at most six steps, their groups are in fact infinite. Our first result extends this result to the remaining models.

Theorem 1.1 *For all 3D models with a group with more than 400 elements, the group is in fact infinite.*

Because of space limitations, and since the proof techniques are exactly the same as in [5, 7], we do not give any further details. We just mention that we used the fixed point method for 10,905,634 models and the valuation method for the 199 models on which the fixed point method failed.

In this short paper, we have a closer look at these infinite groups.

2 Infinite Groups Associated to 3D Models

Recall the definition of the groups [2, 5]. Given $S \subseteq \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$, let $P_{\mathcal{S}}(x, y, z) = \sum_{(i,j,k)\in\mathcal{S}} x^i y^j z^k$. Collecting coefficients of x, y, z, respectively, we can write

$$\begin{aligned} P_{\mathcal{S}}(x, y, z) &= x^{-1}A_-(y, z) + A_0(y, z) + xA_+(y, z) \\ &= y^{-1}B_-(x, z) + B_0(x, z) + yB_+(x, z) \\ &= z^{-1}C_-(x, y) + C_0(x, y) + zC_+(x, y), \end{aligned}$$

for certain bivariate Laurent polynomials $A_-, A_0, A_+, B_-, B_0, B_+, C_-, C_0, C_+$. Then the group of $\mathcal{S}$, denoted by $G(\mathcal{S})$, is generated by the maps

$$\phi_x(x, y, z) = \Big(\frac{A_-}{xA_+}, y, z\Big), \phi_y(x, y, z) = \Big(x, \frac{B_-}{yB_+}, z\Big), \phi_z(x, y, z) = \Big(x, y, \frac{C_-}{zC_+}\Big)$$

under composition. If one of $A_-, A_+, B_-, B_+, C_-, C_+$ is zero, the group is undefined. The stepsets for which this happens are in bijection with lower dimensional models, and are excluded from consideration in this paper.

Example 2.1 The group of $\mathcal{S}_1 = \{(-1, -1, -1), (-1, 1, 1), (1, 0, 1), (1, 1, 0)\}$ is infinite by Theorem 1.1. Another 3D model with infinite group is $\mathcal{S}_2 =$