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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 60 (2017) 83–90 www.elsevier.com/locate/endm

Characterisation of Graphs with Exclusive Sum Labelling

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Abstract

A *sum graph* G is a graph with a mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. In an *exclusive sum graph* the integers of S that are the sum of two other integers of S form a set of integers that label a collection of isolated vertices associated with the graph G. A graph bears a k-*exclusive sum labelling* (abbreviated k-ESL), if the set of isolated vertices is of cardinality k.

In this paper, observing that the property of having a k-ESL is hereditary, we provide a characterisation of graphs that have a k-exclusive sum labelling, for any $k \geq 1$, in terms of describing a universal graph for the property.

Keywords: Graph labelling, exclusive sum graph labelling, hyperdiamond, hereditary property, induced subgraph, universal graph

[1] Research supported by project P202/12/G061 of the Czech Science Foundation.

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http://dx.doi.org/10.1016/j.endm.2017.06.012

1 Introduction

All graphs considered here are simple and undirected unless otherwise stated. All graphs are also connected except for the isolated vertices necessary to maintain the labelling. We will define terms specific to this article, for all other terms used the reader is referred to [2].

A *sum graph* G is a graph with a mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. More formally, for a *sum labelling* $L : V(G) \to S$, we have $u, v \in V(G), uv \in E(G)$, if and only if there is a $w \in V(G)$ such that $L(u) + L(v) = L(w)$. In this case the vertex w is said to be a *working vertex* whose work is to *witness* the edge uv.

Sum graphs were introduced by Harary in [4] as a terse way of storing and communicating graphs. An easy observation is that they must be disconnected; the vertex with the largest label must be an isolate. Any graph can be sum labelled by adding sufficiently many isolated vertices. The *sum number* of a graph G, $\sigma(G)$ is the smallest cardinality of a set of isolates that must be included with G in order for it to have a sum labelling.

A sum graph with all working vertices being confined to the set of isolates was postulated in [7] and given the name *exclusive sum graph*. More precisely, for a given positive integer k, a k*-exclusive sum labelling* (abbreviated k-ESL) of a graph G is a sum labelling L of the graph $G \cup \overline{K_k}$ such that, for $u, v \in V(G \cup \overline{K_k})$, we have $uv \in E(G \cup \overline{K_k})$ if and only if $L(u) + L(v) = L(w)$ for some $w \in \overline{K_k}$ (and, similarly as above, we say that the isolate w *witnesses* the edge uv). We will use $\mathcal{E}_k$ to represent the class of graphs having a k-ESL.

Thus, a (given) k-ESL assigns to every edge of G an isolate by which it is witnessed. This assignment determines an edge colouring of G, in which the colour of an edge equals the label of the isolate by which it is witnessed, and since all labels of vertices have to be distinct, this edge colouring is proper. Moreover, also conversely, once the assignment of labels to the edges of G (i.e., the edge colouring of G) is given, then the labelling L of the vertices of G is uniquely determined, up to an additive constant (provided G is connected; otherwise this is true in each component of G). However, note that not every proper k-edge-colouring of G determines a k-ESL of G: for example, the graph $K_{2,2,2}$ is 4-edge-colourable while any its k-ESL requires $k \geq 7$.

Obviously, if G has a k-ESL, then G has a k'-ESL for every $k' \geq k$. The *exclusive sum number* of a graph G, $\epsilon(G)$ is the smallest k for which G has a k-ESL. Clearly $\sigma(G) \leq \epsilon(G)$ and, by the above observations on edge-colourings,