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Lattice paths with catastrophes

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Abstract

In queuing theory, it is usual to have some models with a "reset" of the queue. In terms of lattice paths, it is like having the possibility of jumping from any altitude to zero. These objects have the interesting feature that they do not have the same intuitive probabilistic behavior like classical Dyck paths (the typical properties of which are strongly related to Brownian motion theory), and this article quantifies some relations between these two types of paths. We give a bijection with some other lattice paths and a link with a continuous fraction expansion, and prove several formulae for related combinatorial structures conjectured in the On-line Encyclopedia of Integer Sequences. Thanks to the kernel method and via analytic combinatorics, we derive the enumeration and limit laws of these "lattice paths with catastrophes", for any finite set of jumps.

Keywords: lattice path, generating function, algebraic function, kernel method, context-free grammar.

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1 Introduction

Lattice paths are a natural model in queuing theory: indeed, the evolution of a queue can be seen as a sum of jumps [11]. In this article, we consider jumps restricted to a given finite set of integers $\mathcal{J}$, where each jump $j \in \mathcal{J}$ is associated with a weight (or probability) p_j. The evolution of a queue naturally corresponds to lattice paths constrained to be non-negative. For example, if $\mathcal{J} = \{-1, +1\}$, this corresponds to the so-called Dyck paths. Moreover, we also consider the model where "catastrophes" are allowed.

Definition 1.1 A *catastrophe* is a jump $j \notin \mathcal{J}$ to altitude 0, see Figure 1.

Such a jump corresponds to a "reset" of the queue. The model of queues with catastrophes was e.g. considered in [14,15].

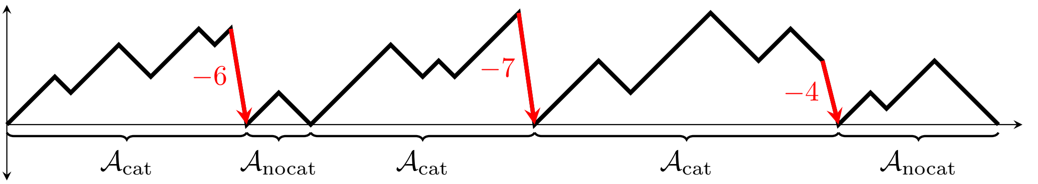

Figure 1. Decomposition of a Dyck path with 3 catastrophes into 5 arches. $\mathcal{A}_{\text{cat}}$ stand for an "arch ending with a catastrophe" (a walk for which the first return is a catastrophe), while $\mathcal{A}_{\text{nocat}}$ stands for an "arch with no catastrophe".

Link with a continuous fraction expansion. We first start with the observation that the generating function of these lattice paths have the following continuous fraction expansion:

$$H(z) = \sum_{n \geq 0} h_n z^n = \cfrac{1}{1 - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \ddots}}}}}.$$

(We give two proofs of this phenomenon in Theorem 3.1). In this article, we also tackle the question of what happens for more general jumps than Dyck paths, and we give the enumeration and asymptotics of the corresponding lattice path models.