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Proper connection number 2, connectivity, and forbidden subgraphs

Christoph Brause[a], Trung Duy Doan[a,b,1], Ingo Schiermeyer[a]

[a] *Institute of Discrete Mathematics and Algebra*
TU Bergakademie Freiberg
Freiberg, Germany
brause@math.tu-freiberg.de, ingo.schiermeyer@tu-freiberg.de

[b] *School of Applied Mathematics and Informatics*
Hanoi University of Science and Technology
Hanoi, Vietnam
trungdoanduy@gmail.com

Abstract

An edge-coloured graph G is called *properly connected* if any two vertices are connected by a path whose edges are properly coloured. The *proper connection number* of a graph G, denoted by $pc(G)$, is the smallest number of colours that are needed in order to make G properly connected. In this paper we consider sufficient conditions in terms of connectivity and forbidden subgraphs, implying a graph to have proper connection number 2.

Keywords: proper connection number, 2-connected, forbidden subgraphs

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1 Introduction

We use [4] for terminology and notation not defined here and consider simple and undirected graphs only.

The concept of proper connections in graphs is an extension of proper colourings and is motivated by rainbow connections of graphs. Andrews et al. [1] and, independently, Borozan et al. [3] introduced the concept as follows:

An edge-coloured graph G is called *properly connected* if every two vertices $u, v \in V(G)$ are connected by a path whose edges are properly coloured. The proper connection number $pc(G)$ is the smallest number of colours needed to colour a graph G properly connected. We say, an edge-colouring c has the *strong property* if for every two vertices $u, v \in V(G)$ there exists two properly coloured paths $P_1 : u = w_1w_2 \dots w_k = v$ and $P_2 : u = z_1z_2 \dots z_l = v$ such that $c(w_1w_2) \neq c(z_1z_2)$ and $c(w_{k-1}w_k) \neq c(z_{l-1}z_l)$. We note that $pc(G) = 1$ if and only if G is complete [3].

For simplifying notation, let $[k]$ be the set $\{1, 2, \dots, k\}$ for some positive integer k. Following common notation, we say G contains an *induced subgraph* F if there is a vertex subset $U \subseteq V(G)$ such that $G[U] \cong F$. Therefore, G is F*-free* ($\mathcal{F}$-free) if and only if G contains F (all graphs of $\mathcal{F}$) not as an induced subgraph. Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths i,j, and k with a common initial vertex, and $\mathcal{S}$ be the set of graphs whose every component is of the form $S_{i,j,k}$ for some $0 \leq i \leq j \leq k$.

In many fields of graph theory, forbidden subgraphs and the connectivity of a graph play an important role. In [2], Bedrossian characterized pairs of forbidden subgraphs for 2-connected graphs implying hamiltonicity. Thus, since every noncomplete, hamiltonian graph has proper connection number 2 [3], his characterization is the starting point for our work to find sufficient conditions in terms of connectivity and forbidden subgraphs such that $pc(G) = 2$ holds for a graph G. We note that all pairs in Bedrossian's characterization contain the claw. Our first result improves that observation by forbidding only the claw.

Theorem 1.1 *Let G be a connected, claw-free, and noncomplete graph. Then $pc(G) = 2$.*

Sketch of the Proof. Suppose, to the contrary, that there exists a connected, claw-free graph of proper connection number at least 3. Moreover, all those graphs are noncomplete. Then, let G be a counterexample of minimum order, i.e. G is connected, claw-free, but $pc(G) \geq 3$, and for all noncomplete but connected induced subgraphs G' of G, it holds $pc(G') = 2$. Now, let H be a