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Subset matching and edge coloring in bipartite graphs

Ömer Can Yavuzyılmaz*, Enver Kayaaslan

Turgut Ozal University, Ankara, Turkey
omercanyy@gmail.com, ekayaaslan@turgutozal.edu.tr

Abstract

The focus of this paper is on finding a matching in a bipartite graph such that a given subset of vertices are matched. This is called subset matching and generalizes perfect matchings. We prove a necessary and sufficient condition for the existence of a subset matching in bipartite graphs. The proof is algorithmic and based on combination of two matchings. Remarkably, the necessary and sufficient condition always holds when the subset is composed of the vertices with maximum degree. This in turn leads to a simple algorithm that finds an optimal edge coloring in bipartite graphs with no need to transform the bipartite graph into a regular one.

Keywords: bipartite graph, matching, edge coloring.

Bipartite graphs represent the class of (undirected) graphs without odd cycles, and a matching is a subgraph in which every vertex is incident to at most one edge. Throughout the presentation of our findings, we assume that the reader is familiar with bipartite graphs and matchings [10]. A closely related study is by Alon and Yuster [2], which introduces maximum subset matching in general graphs and gives an approximation algorithm. In our study, we are interested in subset matching in bipartite graphs, where a given subset of vertices are all matched. Notice that a subset matching is, in fact, a perfect matching for the bipartite graph when the subset contains all vertices.

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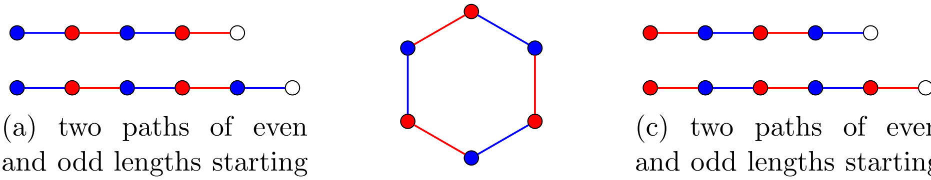

(a) two paths of even and odd lengths starting from the blue vertices.

(b) a cycle.

(c) two paths of even and odd lengths starting from the red vertices.

Fig. 1. Possible components of the union graph of two matchings.

Problem 1 Subset matching in bipartite graphs *Given a bipartite graph* $\mathcal{B}(\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ *and a subset* $S \subset \mathcal{V}_1 \cup \mathcal{V}_2$ *of vertices, a matching* $\mathcal{M}$ *is called a subset matching for* S *if there is an edge* $\{u, v\} \in \mathcal{M}$ *for every* $v \in S$.

The following theorem presents a necessary and sufficient condition for a bipartite graph to have a subset matching for a given subset of vertices.

Theorem 1 Necessary and sufficient condition *Given a bipartite graph* $\mathcal{B}(\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ *and* $S \subset \mathcal{V}_1 \cup \mathcal{V}_2$*, there is a subset matching* $\mathcal{M}$ *for* S *if and only if* $|X_i| \leq |\mathcal{N}(X_i)|$ *for any* $X_i \subset S \cap \mathcal{V}_i$*, for each* $i = 1, 2$.

Proof. Hall's Theorem [6] implies that there exists a matching that covers $S \cap \mathcal{V}_i$ if and only if $|X_i| \leq |\mathcal{N}(X_i)|$ for any $X_i \subset S \cap \mathcal{V}_i$, for each $i = 1, 2$.

Assume that there is a subset matching $\mathcal{M}$ for S. Then, $\mathcal{M}$ covers both $S \cap \mathcal{V}_1$ and $S \cap \mathcal{V}_2$, and due to Hall's Theorem, $|X_i| \leq |\mathcal{N}(X_i)|$ for any $X_i \subset S \cap \mathcal{V}_i$, for each $i = 1, 2$.

Assume that $|X_i| \leq |\mathcal{N}(X_i)|$ for any $X_i \subset S \cap \mathcal{V}_i$, for each $i = 1, 2$. By Hall's Theorem, there are two matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ that cover $S \cap \mathcal{V}_1$ and $S \cap \mathcal{V}_2$, respectively. Now, we show that there is a subset matching $\mathcal{M}$ for S. Consider the union graph $\mathcal{M}_1 \cup \mathcal{M}_2$, which is also a subgraph of the bipartite graph. We categorize vertices into three groups: the vertices of $S \cap \mathcal{V}_1$ (shown as blue circles in Fig. 1), the vertices of $S \cap \mathcal{V}_2$ (red circles), and the vertices of $\mathcal{V}_1 \cup \mathcal{V}_2 - S$ (empty circles). Notice that any vertex in the union graph is either isolated, or incident to exactly one edge, or incident to exactly two edges (one from $\mathcal{M}_1$ and the other from $\mathcal{M}_2$). Thus a connected component of the union graph is either a path, or a cycle of even length [9,11,4], as depicted in Fig. 1, where the blue and red edges represent edges of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. For a path that starts from a (blue) vertex in $S \cap \mathcal{V}_1$ (Fig. 1a), we pick the corresponding (blue) edges of $\mathcal{M}_1$ to be included in $\mathcal{M}$. For a cycle (Fig. 1b), we pick either the corresponding (blue) edges of $\mathcal{M}_1$ or (red) edges of $\mathcal{M}_2$ for $\mathcal{M}$. Finally, for a path that starts from a (red) vertex in $S \cap \mathcal{V}_2$ (Fig. 1c), we pick the corresponding (red) edges of $\mathcal{M}_2$ to be included in $\mathcal{M}$. Then, $\mathcal{M}$ is a matching and covers all vertices of S. Thus, $\mathcal{M}$ is a subset matching for S. $\square$