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Weighted Graphs as Dynamical Interaction Systems

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Abstract

The article shows that the common models for cooperation and interaction may naturally be viewed in a quantum context when the relevant data structures are represented according to their symmetry decomposition.

Keywords: Graph, interaction system, Markov evolution, Schrödinger evolution

1 Symmetric interaction systems

A *(weighted) graph* on $N = \{1, \ldots, n\}$ is a map $A : N \times N \to \mathbb{R}$ and thus represented by a real $n \times n$ matrix $A = [A_{ij}]$. If N is a set of agents, A corresponds to a situation of pairwise interaction in N with structural coefficients A_{ij}. The vector space $\mathbb{R}^{n \times n}$ of all real $n \times n$ matrices is n^2-dimensional is a (real) Hilbert space with the inner product

$$\langle A|B\rangle = \sum_{ij} A_{ij}B_{ij} = \mathrm{tr}(B^T A) \quad \text{and norm} \quad \|A\| = \sqrt{\langle A|A\rangle}.$$

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A *simple* (real) interaction A is described by a vector $a \in \mathbb{R}^n$ of normed euclidian length $\|a\| = 1$ and an *amplitude* $\lambda \in \mathbb{R}$ such that

$$A_{ij} = \lambda a_i a_j \ \forall i, j \in N\ , \quad \text{i.e.,}\ \ A = \lambda \cdot aa^T.$$

NOTE: The n diagonal elements a_i^2 of aa^T define a probability distribution on N.

Theorem 1.1 (Spectral decomposition) *$A \in \mathbb{R}^{n\times n}$ is symmetric if and only if A is the superposition of n pairwise orthogonal simple interactions $\lambda_i P_i$:*

$$A = \sum_{i\in N} \lambda_i P_i.$$

The amplitudes λ_i are the eigenvalues of A. $\diamond$

The (well-known) spectral decomposition (Theorem 1.1) shows that the dynamics of symmetric interaction systems are determined by the dynamics of simple interactions. Note furthermore,

$$A^t = \sum_{i\in N} \lambda_i^t P_i \quad (t = 1, 2, \ldots)$$

2 Markov and Schrödinger evolution

Let $A = \lambda P$ with $P = aa^T$ be a simple interaction with amplitude λ. The *Markov evolution* is based on the evolution of the amplitude, *i.e.*, on the repeated application of A and thus the *Markov chain*

$$A^2 = \lambda^2 P, A^3 = \lambda^3 P, \ldots, A^t = \lambda^t P, \ldots$$

So the evolution converges to a limit if and only if the amplitudes λ^t converge to a limit.

In the *Schrödinger picture* the evolution of $A = \lambda P$ is described by an orthogonal transformation $a \mapsto Ua$ of $\mathbb{R}^n$ (with $U^{-1} = U^T$) and hence the chain

$$\lambda(Ua)(Ua)^T = \lambda UPU^T, \ldots, \lambda(U^t a)(U^t a)^T = \lambda(UPU^T)^t, \ldots.$$

of simple interactions with identical amplitude λ.

NOTE: In both evolution models, the evolution is derived from a linear operator on the space of all graphs (interactions).