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Small doubling in ordered nilpotent groups of class 2

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ABSTRACT

The aim of this paper is to present a complete description of the structure of finite subsets S of torsion-free nilpotent groups of class 2 satisfying $|S^2| = 3|S| - 2$. In view of results in [12], this gives a complete description of the structure of finite subsets with the above property in any torsion-free nilpotent group.

1. Introduction

Let α and β denote real numbers, with $\alpha > 1$. A finite subset S of a group G is said to satisfy the *small doubling property* if

$$|S^2| \leq \alpha|S| + \beta \,,$$

where $S^2 = \{s_1s_2 \mid s_1, s_2 \in S\}$.

The classical Freiman's inverse theorems describe the structure of finite subsets of abelian groups, which satisfy the small doubling property (see [6–9,19] and [22]). Recently, several authors obtained similar results concerning various classes of groups for an arbitrary α (see for example [2–5,10,15–17,21,23,24] and [25]).

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In particular E. Breuillard and B. Green in [3] and M. Tointon in [25] investigated the problem in the case of G being a nilpotent group.

In [10] we started the investigation of finite subsets of *ordered groups* satisfying the small doubling property with $\alpha = 3$ and small $|\beta|$. We proved that if $(G, <)$ is an ordered group and S is a finite subset of G of size $k \geq 2$, such that $|S^2| \leq 3k - 3$, then $\langle S\rangle$ is abelian. Furthermore, if $k \geq 3$ and $|S^2| \leq 3k-4$, then there exist $x_1, g \in G$ such that $g > 1, gx_1 = x_1g$ and S is a subset of the geometric progression $\{x_1, x_1g, x_1g^2, \dots, x_1g^{t-k}\}$, where $t = |S^2|$. We also showed that these results are best possible, by presenting an example of an ordered group with a subset S of size k with $\langle S\rangle$ non-abelian and $|S^2| = 3k - 2$.

Other recent results concerning small doubling properties appear in [11–14].

In [12] we studied finite subsets S of torsion-free nilpotent groups such that $|S^2| = 3k - 2$.

The aim of the present paper is to give a complete description of the structure of subsets S of size k in torsion-free nilpotent groups satisfying $|S^2| = 3k - 2$.

It is known that torsion-free nilpotent groups are orderable (see [18] and [20]), so the results in [10] apply. Furthermore, we can assume that the class of G is 2, because of Theorem 5 and Corollary 3 in [12].

Our main result is the following theorem.

Theorem 1.1. *Let G be a torsion-free nilpotent group and let S be a subset of G of size $k \geq 4$ with $\langle S\rangle$ non-abelian. Then $|S^2| = 3k - 2$ if and only if there exist $a, b, c \in G$ and non-negative integers i, j such that*

$$S = \{a, ac, \dots, ac^i, b, bc, \dots, bc^j\},$$

with $1 + i + 1 + j = k$, $c \neq 1$ and $[a, b] = c^{\pm 1}$.

Here $[a, b] = a^{-1}b^{-1}ab$ is the commutator of the elements a, b.

We refer to [1] and [18] for results concerning ordered groups. In particular, we use at several places the following result proved by B.H. Neumann in [20].

Proposition. *If G is an ordered group and $a, b \in G$ are such that a commutes with b^n for some integer $n \geq 1$, then a commutes with b.*

2. Some general results

We start with the following two very useful lemmas.

Lemma 2.1. *Let $(G, <)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 3$, satisfying*

$$S = \{x_1, \dots, x_k\}, \quad x_1 < x_2 < \dots < x_k.$$

Let $T = \{x_1, \dots, x_{k-1}\}$ and for any positive $j \leq k - 1$, let $T_j = \{x_j, \dots, x_{k-1}\}$. Suppose that for some positive $i \leq k - 2$, we have

$$\{x_ix_k, x_kx_i\} \subseteq T^2.$$

Then either x_ix_k or x_kx_i belongs to T^2_{i+1}.

Proof. Write $x_ix_k = x_jx_l$, for suitable $l, j \leq k - 1$, then $j \geq i + 1$. Similarly, write $x_kx_i = x_sx_t$, for some $s, t \leq k - 1$, then $t \geq i + 1$. We claim that either $l \geq i + 1$ or $s \geq i + 1$. We have $x_ix_k = x_kx_i[x_i, x_k]$ and either $[x_i, x_k] \geq 1$ or $[x_i, x_k] \leq 1$. Suppose, for example, that $[x_i, x_k] \geq 1$. Then $[x_k, x_i] \leq 1$, and $x_sx_t = x_kx_i = x_ix_k[x_k, x_i] \leq x_ix_k$, thus, from $t < k$ it follows that $s \geq i + 1$. Similarly, if $[x_i, x_k] \leq 1$, we get $l \geq i + 1$. □