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Decomposition of tournament limits

Erik Thörnblad

Department of Mathematics, Uppsala University, Box 480, S-75106 Uppsala, Sweden

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ABSTRACT

The theory of tournament limits and tournament kernels is developed by extending common notions for finite tournaments to this setting; in particular we study transitivity and irreducibility of limits and kernels. We prove that each tournament kernel and each tournament limit can be decomposed into a direct sum of irreducible components, with transitive components interlaced. We also show that this decomposition is essentially unique.

1. Introduction

Informally speaking, graph limits are abstract limit objects of graph sequences $(G_n)_{n=1}^{\infty}$ with $v(G_n) \to \infty$, all of whose associated subgraph densities converge. This theory was initiated for undirected graphs in [19] and later developed in among others [4–6]. The limit objects can be non-trivial only if $|E(G_n)| = \Theta(v(G_n)^2)$, so in this sense the theory of graph limits concerns itself with sequences of dense graphs, although attempts have been made to extend this notion to the sparse setting, see e.g. [18], which also provides a general overview of the theory of graph limits. For dense graph limits, the limit of a sequence of graphs is not sensitive to perturbations. In fact, if $(G_n)_{n=1}^{\infty}$ is a sequence of graphs converging to some graph limit, it can be shown that the sequence of graphs, obtained by removing or adding up to $o(v(G_n)^2)$ edges, will still converge to the same graph limit. It is common to try to extend results known about finite graphs to the setting of graph limits, and the non-sensitivity to perturbations of the sequence, coupled with additional analytical tools that become available, often makes results easier to prove in the limit case. However, it should be mentioned that it is not always the case that one can carry out a sensible extension of the theory of finite graphs to graph limits.

In the undirected case, each graph limit can be represented by a so-called *kernel* or *graphon*, which is a symmetric function $[0, 1]^2 \to [0, 1]$. The correspondence between kernels and the graph limits they represent is highly-nontrivial, and in general there are many kernels representing the same

E-mail address: erik.thornblad@math.uu.se.

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graph limit. The *cut norm* defined on the set of such symmetric functions provides a framework of determining when two kernels represent the same graph limit. We shall avoid using the cut norm in this paper, so we do not pursue this matter further here.

There is a also connection between graph limit theory and probability theory, demonstrated (independently) in [1] and [9]. The connection is that graph limits appear as extreme points in the set of distributions of exchangeable arrays. For directed graphs, a consequence is that each directed graph limit can be represented by a kernel, which turns out to be a quintuple of functions. For undirected graphs, a single function suffices. Another case where the kernels are of a simpler for are for tournaments. A tournament is a directed graph with precisely one edge between any two vertices. The (tournament) kernels corresponding to tournament limits have a particularly easy form; indeed, we shall see later that it suffices to consider functions $W : [0, 1]^2 \to [0, 1]$ satisfying $W(x, y)+W(y, x) = 1$. A trivial fact about tournaments is that these are dense directed graphs, so we can expect non-trivial limits and kernels of tournament sequences to appear.

Although we will develop the general framework of tournament limits and extend the notions of irreducibility and transitivity of finite tournaments to the setting of tournament kernels and tournament limits, our main aim is a certain decomposition result. Decomposition of finite graphs into 'components' is a well-studied problem. The paper [7] contains several decomposition results for both undirected and directed graphs, both in the finite and infinite case. Generally speaking, a decomposition of a graph is a partition of its edge set or its vertex sets into subsets, typically under the requirement that each part of the partition satisfies some desired property (by themselves or pairwise). The simplest result of this type is perhaps the decomposition of a finite graph into its connected components, which means a partition of the vertex set such that each part induces a connected subgraph, and there are no edges between different components. This type of decomposition was extended to (undirected) graph kernels and graph limits by Janson [14].

To be precise, our aim is to extend the following decomposition result mentioned by Moon [22] for finite tournaments. Given any tournament, its vertex set can be uniquely partitioned into subsets, each of which induces either an irreducible tournament (there are directed paths in both directions between any pair of vertices) or a transitive tournament (there are no cyclic subgraphs), such that these components can be linearly ordered with direction of the edges between the components respecting the linear ordering.

In the following paragraphs we outline the remainder of the paper.

In Section 2, we give the necessary background of directed graph limit theory. We outline how subgraph densities form the basis of the theory and the connection between our main objects – digraph limits, digraph kernels and their associated (infinite) random graphs.

In Section 3, we collect the necessary background on tournament theory. We give some basic results about finite and countable tournaments, most of which are known. First, we introduce transitive tournaments, which are tournaments which do not contain any directed cycles. Then, we consider irreducible tournaments, which are tournaments whose vertex set cannot be split into two disjoint sets, such that all edges between the sets go from one of the sets to the other. Transitivity and irreducibility are two of the key notions in tournament theory, and both properties can be characterised in several different way; see Theorems 3.3 and 3.6. Finally, we show that each (countable) tournament has an essentially unique decomposition into irreducible subtournaments. This extends (to countable tournaments) the classical decomposition result [22] mentioned above.

In Section 4, we begin our study of tournament limits. A consequence of Theorem 4.1 is that each tournament limit can be represented by a single function W such that $W(x, y) + W(y, x) = 1$. (By analogy to the adjacency matrix of tournaments, this is precisely what is to be expected). These functions are our tournament kernels.

In Section 5, we extend the notion of transitivity to tournament limits and tournament kernels. Like for finite tournaments, transitive tournament kernels have many different representations, see Theorem 5.4. Just like there is a unique (up to isomorphism) transitive tournament of each given size, there is essentially only one transitive tournament limit, given by $W(x, y) = \mathbb{1}_{\{x \le y\}}$.

In the remaining sections, our main aim is to extend the decomposition result discussed in Section 3 for countable tournaments. These ideas draw heavily on [14]. In Section 6, we do this for tournament kernels, by first defining direct sums of tournament limits, and then by showing that each tournament