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Modular flip-graphs of one-holed surfaces

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ABSTRACT

We study flip-graphs of triangulations on topological surfaces where distance is measured by counting the number of necessary flip operations between two triangulations. We focus on surfaces of positive genus g with a single boundary curve and n marked points on this curve and consider triangulations up to homeomorphism with the marked points as their vertices. Our results are bounds on the maximal distance between two triangulations. Our lower bounds assert that these distances grow at least like $5n/2$ for all $g \geq 1$. Our upper bounds grow at most like $[4 - 1/(4g)]n$ for $g \geq 2$, and at most like $23n/8$ for the bordered torus.

1. Introduction

The set of triangulations of a given surface can be given a geometry by defining the distance between two triangulations as the necessary number of flip operations needed to transform one of them into the other. As stated, this definition is somewhat vague, but if the surface is a (Euclidean) polygon and we think of triangulations as being geometric (realized by line segments), then a flip consists in removing an edge from a triangulation and replacing it by the unique other edge so that the result is still a triangulation of the polygon. Given this geometry, the set of triangulations of a polygon is well studied: it is the graph of the associahedron [4,10–12]. In particular its diameter is $2n - 10$ for $n > 12$ [8,9]. Such graphs, called *flip-graphs* arise in a number of other, yet related settings [1,5] and, when they are connected, finding their diameter is often notoriously difficult.

The graph of the associahedron can be defined in purely topological terms: a polygon is a topological disk with n marked points on its boundary and a triangulation is a maximal collection of disjoint isotopy classes of arcs whose endpoints are among the marked points. (The isotopies are relative to the endpoints and by disjoint it is meant disjoint in their interiors.) Flip transformations

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are topological; one way of defining them is that two triangulations are related by a flip if they differ only in one edge. This topological description works when one replaces the disk by any finite type surface with marked points, provided there are marked points on each boundary curve. However, provided the surface has enough topology (for instance if it has positive genus), the flip-graph is infinite and has infinite diameter. The mapping class group (the self-homeomorphisms up to isotopy) of the underlying surface acts nicely on it: indeed it is basically the isomorphism group of the graph (see Section 2 for precise definitions, statements, and references). As such, we can quotient flip-graphs to obtain well-defined finite graphs whose vertices represent all possible types of triangulations and whose diameter can be measured. We call these graphs *modular flip-graphs* and we are interested in their geometry.

In a previous paper [7], we explored the diameter of the modular flip-graphs of *filling surfaces*: these surfaces have a privileged boundary curve, but otherwise arbitrary topology. In particular, they were allowed to have interior marked points, more than one boundary, and arbitrary genus. Here we focus our attention to the special case of *one-holed surfaces* that have a unique boundary curve and no interior marked points. We investigate the growth of the diameter of the corresponding modular flip-graphs in function of the number of marked points on the boundary, while the genus is fixed. These points are *labeled*: we quotient by homeomorphisms that leave them individually fixed.

The case that we focus on most is the torus; it provides a natural variant on the case of the disk and, as we shall see, is already quite intriguing. For this surface we are able to prove the following.

Theorem 1. *Let Σ_n be a one-holed torus with n marked points on the boundary and let $\mathcal{MF}(\Sigma_n)$ be its modular flip-graph. Then*

$$\left\lfloor \frac{5}{2}n \right\rfloor - 2 \leq \mathrm{diam}(\mathcal{MF}(\Sigma_n)) \leq \frac{23}{8}n + 8.$$

There is a clear gap between our upper and lower bounds (on the order of $3n/8$ for large n) that we are unable to close; in fact it is even tricky to guess what the correct growth rate might be. We point out that, in the instances where matching upper and lower bounds are known [7–9] the lower bounds have always been the more difficult ones to obtain.

The methods used to prove the above theorem also provide more general results about surfaces of arbitrary genus $g \geq 1$.

Theorem 2. *Let Σ_n be a one-holed surface of genus $g \geq 1$ with n marked points on the boundary and let $\mathcal{MF}(\Sigma_n)$ be its modular flip-graph. Then*

$$\left\lfloor \frac{5}{2}n \right\rfloor - 2 \leq \mathrm{diam}(\mathcal{MF}(\Sigma_n)) \leq \left(4 - \frac{1}{4g}\right) n + K_g,$$

where K_g only depends on g.

As a direct consequence of this result and the results of [7], we obtain the following.

Corollary 1. *Let Σ_n be a filling surface with fixed topology and n marked points on the privileged boundary. If this surface is not homeomorphic to a disk or a once-punctured disk, then the diameter of its modular flip-graph grows at least as $5n/2$.*

2. Preliminaries

We begin this section by defining and describing the objects we are interested in.

Our basic setup is as follows. Consider an orientable topological surface Σ of finite type with a single boundary curve. It has no marked points on it but will be endowed with them in what follows. We will denote by $g \geq 0$ the genus of Σ (so if $g = 0$ then Σ is a disk).

For any positive integer n, from Σ we obtain a surface Σ_n by placing n marked points on the boundary of Σ, that we refer to as a *one-holed surface*. These marked points are *labeled* from a_1 to a_n, clockwise around the boundary. We are interested in triangulating Σ_n and studying the geometry