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On sequences of polynomials arising from graph invariants

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A R T I C L E I N F O

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A B S T R A C T

Graph polynomials are deemed useful if they give rise to algebraic characterizations of various graph properties, and their evaluations encode many other graph invariants. Algebraic: The complete graphs K_n and the complete bipartite graphs $K_{n,n}$ can be characterized as those graphs whose matching polynomials satisfy a certain recurrence relations and are related to the Hermite and Laguerre polynomials. An encoded graph invariant: The absolute value of the chromatic polynomial $\chi(G, X)$ of a graph G evaluated at -1 counts the number of acyclic orientations of G.

In this paper we prove a general theorem on graph families which are characterized by families of polynomials satisfying linear recurrence relations. This gives infinitely many instances similar to the characterization of $K_{n,n}$. We also show where to use, instead of the Hermite and Laguerre polynomials, linear recurrence relations where the coefficients do not depend on n.

Finally, we discuss the distinctive power of graph polynomials in specific form.

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1. Introduction and background

1.1. Wilf's recognition problem

H. Wilf asked in [37] to characterize and recognize the instances of the chromatic polynomial. C.D. Godsil and I. Gutman [15] gave a characterization of the instances of the defect matching polynomial $\mu(G; X)$ for paths P_n, cycles C_n, complete graphs K_n and bipartite complete graphs $K_{n,n}$ in terms of orthogonal polynomials. We want to put Wilf's question and C.D. Godsil and I. Gutman's observation into a larger perspective. First we have to fix some terminology. Let $\mathcal{G}$ denote the class of all finite graphs with no multiple edges. A graph property is a class of graphs $\mathcal{C} \subseteq \mathcal{G}$ closed under graph isomorphism. A graph parameter $f(G)$ is a function $\mathcal{G} \to \mathbb{Z}$ invariant under graph isomorphism. A graph polynomial with r indeterminates[4] $\bar{X} = (X_1, \ldots, X_r)$ is a function $\mathbf{P}$ from all finite graphs into the polynomial ring $\mathbb{Z}[\bar{X}]$ which is invariant under graph isomorphism. We write $\mathbf{P}(G; \bar{X})$ for the polynomial associated with the graph G.

Definition 1. A graph polynomial $\mathbf{P}$ *is computable* if

(i) $\mathbf{P}$ is a Turing computable function, and additionally,
(ii) the range of $\mathbf{P}$, the set

$$\{p(\bar{X}) \in \mathbb{Z}[\bar{X}] : \text{ there is a graph } G \text{ with } \mathbf{P}(G; \bar{X}) = p(\bar{X})\}$$

is Turing decidable.

Definition 2. A graph polynomial $\mathbf{P}$ *is bounded* if there is a recursive function $\beta_{\mathbf{P}} : \mathbb{N} \to \mathbb{N}$ such that for each $p(\bar{X}) \in \mathcal{R}[\bar{X}]$ with total degree $d = d(p(\bar{X}))$, if there is at all a graph G with $\mathbf{P}(G, \bar{X}) = p(\bar{X})$, there is also a graph G' with $\mathbf{P}(G', \bar{X}) = p(\bar{X})$ of order $\leq \beta_{\mathbf{P}}(d)$.

All the graph polynomials encountered in the literature are bounded.
In this paper we give a general formulation to Wilf's question.

Problem 1 (*Recognition and Characterization Problem*)**.** Given a graph polynomial $\mathbf{P}(G; \bar{X})$ and a graph property $\mathcal{C}$, define

$$\mathcal{Y}_{\mathbf{P},\mathcal{C}} = \{p(\bar{X}) \in \mathbb{Z}[\bar{X}] : \exists G \in \mathcal{C} \text{ with } \mathbf{P}(G; \bar{X}) = p(\bar{X})\}$$

(i) The *recognition problem* asks for an algebraic method to decide membership in $\mathcal{Y}_{\mathbf{P},\mathcal{C}}$.
(ii) The *characterization problem* asks for an *algebraic characterization of* $\mathcal{Y}_{\mathbf{P},\mathcal{C}}$, i.e., an algebraic characterization of the coefficient sequence of $p(\bar{X})$.

Both the recognition and the characterization problem were stated explicitly for the chromatic polynomial $\chi(G; X)$ and $\mathcal{C}$ the class of all finite graphs by H. Wilf, [37], and he deemed them to be very difficult.

When H. Wilf asked the question about the chromatic polynomial he had an algebraic and descriptive answer in mind. Something like, a polynomial $p(X)$ is a chromatic polynomial of some graph G iff the coefficients satisfy some relations. The conjecture, that the absolute values of the coefficients of the chromatic polynomial form a unimodal sequence, only recently proved by J. Huh, [20] has its origin in Wilf's question. H. Wilf was not concerned about algorithmic complexity.

From a complexity point of view, we note that deciding whether a given polynomial $p(X)$ is a chromatic polynomial of a some graph G can be decided by brute force in exponential time as follows:

(i) Use the degree d_p of $p(X)$ to determine the upper bound on the size of the candidate graph G. In the case of the chromatic polynomial we have $|V(G)| = d_p$.
(ii) Let $I(n)$ be the number of graphs, up to isomorphism, of order n. Listing all graphs, up to isomorphism, of order n, is exponential in n.

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[4] If the polynomial is univariate, we write X instead of $\bar{X}$.