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Decomposing regular graphs with prescribed girth into paths of given length

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A B S T R A C T

A P_ℓ-decomposition of a graph G is a set of pairwise edge-disjoint paths with ℓ edges that cover the edge set of G. In 1957, Kotzig proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching, and also asked what are the necessary and sufficient conditions for an ℓ-regular graph to admit a P_ℓ-decomposition, for odd ℓ. Let g, ℓ and m be positive integers with $g \geq 3$. We prove that, (i) if ℓ is odd and $m > 2\lfloor(\ell-2)/(g-2)\rfloor$, then every $m\ell$-regular graph with girth at least g that contains an m-factor admits a P_ℓ-decomposition; (ii) if $m > \lfloor(\ell-2)/(g-2)\rfloor$, then every $2m\ell$-regular graph with girth at least g admits a P_ℓ-decomposition. Furthermore, we prove that, for graphs with girth at least $\ell-1$, statement (i) holds for every $m \geq 1$; and observe that, statement (ii) also holds for every $m \geq 1$.

1. Introduction

A set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of pairwise edge-disjoint subgraphs of a graph G is called a *decomposition* of G if these subgraphs cover the edge set of G. If $H_1, \ldots, H_k$ are all isomorphic to a graph H, then we say that $\mathcal{D}$ is an H-*decomposition* of G. In this paper we focus on the special case where G is a regular graph, and H is a path with ℓ edges, which we denote by P_ℓ.

Kotzig [15] (see also Bouchet and Fouquet [6]) proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching. Kotzig asked what are the necessary and sufficient conditions for an ℓ-regular graph G, with odd ℓ, to admit a P_ℓ-decomposition. For $\ell = 5$, the following sufficient conditions have been proved. Favaron, Genest and Kouider [9] proved that it suffices that G contain a perfect matching and no cycle of length four. And recently, it was proved [4] that it suffices that G contain a perfect matching and no triangle.

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Let $\mathcal{D}$ be a decomposition of a graph G into trails. Given a vertex v of G, we denote by $\mathcal{D}(v)$ the number of elements of $\mathcal{D}$ containing v as an end-vertex. We say that $\mathcal{D}$ is *balanced* if $\mathcal{D}(u) = \mathcal{D}(v)$ for every $u, v \in V(G)$. Heinrich, Liu and Yu [12] proved that if G is a $3m$-regular graph that contains an m-factor, then G admits a balanced P_3-decomposition.

Kouider and Lonc [16] proved that if G is a 2ℓ-regular graph with girth at least $(\ell + 3)/2$, then G admits a balanced P_ℓ-decomposition. They also proved that if ℓ is even, then every ℓ-regular bipartite graph with girth at least $(\ell + 3)/2$ admits a P_ℓ-decomposition. Moreover, they proposed the following conjecture.

Conjecture 1.1. *Every 2ℓ-regular graph admits a balanced P_ℓ-decomposition.*

Thus, to settle Conjecture 1.1, it remains to verify it for graphs with girth smaller than $(\ell + 3)/2$. In fact, Conjecture 1.1 is related to a conjecture concerning decompositions of 2ℓ-regular graphs into trees with ℓ edges posed by Häggkvist [11] (the reader may refer to [8,10,13,14,18] for more results on decomposition of regular graphs into trees).

The following conjecture, of a similar nature, but concerning odd regular graphs, was proposed by Favaron, Genest and Kouider [9].

Conjecture 1.2. *Let ℓ be an odd integer. If G is an ℓ-regular graph that contains a perfect matching, then G admits a balanced P_ℓ-decomposition.*

We consider the problem of obtaining balanced P_ℓ-decompositions of $m\ell$-regular graphs. We propose the following conjecture, which is a strengthening of Conjecture 1.2 (see Conjecture 1.3(i)) and an equivalent form of Conjecture 1.1 (see Conjecture 1.3(ii)). The equivalence of Conjectures 1.1 and 1.3(ii) follows from Petersen's Factorization Theorem (see Theorem 2.7).

Conjecture 1.3. *Let m and ℓ be positive integers. Then, the following holds.*

(i) *If ℓ is odd, then every $m\ell$-regular graph that contains an m-factor admits a balanced P_ℓ-decomposition.*
(ii) *Every $2m\ell$-regular graph admits a balanced P_ℓ-decomposition.*

Note that Conjecture 1.3(ii) is false if, instead of $2m\ell$, we consider $m\ell$ with m odd and ℓ even. Indeed, if G is an $m\ell$-regular graph on n vertices, then $|E(G)| = nm\ell/2$. If G admits a P_ℓ-decomposition $\mathcal{D}$, then $|\mathcal{D}| = nm/2$. Also, $|\mathcal{D}| = \frac{1}{2}\sum_{v\in V(G)}\mathcal{D}(v)$. If $\mathcal{D}$ is balanced, then $|\mathcal{D}| = n\mathcal{D}(v)/2$ for any $v \in V(G)$. Therefore, $n\mathcal{D}(v)/2 = nm/2$, hence $\mathcal{D}(v) = m$ for every $v \in V(G)$. But since $\mathcal{D}$ is a path decomposition and G is Eulerian, $\mathcal{D}(v)$ must be even for any vertex $v \in V(G)$, a contradiction.

The basic terminology and notation used in this paper are standard [1,7]. All graphs considered here are simple, and therefore have girth at least 3. Throughout this paper, the symbols g, ℓ, k, m and r denote positive integers.

This paper is organized as follows. Section 2 is devoted to the proofs of our main results, Theorems 2.8 and 2.9. We prove that Conjecture 1.3(i) holds for $m\ell$-regular graphs with girth at least g and $m > 2\lfloor(\ell - 2)/(g - 2)\rfloor$; and Conjecture 1.3(ii) holds for $2m\ell$-regular graphs with girth at least g and $m > \lfloor(\ell - 2)/(g - 2)\rfloor$. In particular, these results show that Conjecture 1.3(i) holds when $m \geq 2\ell - 3$; and Conjecture 1.3(ii) holds when $m \geq \ell - 1$. In Section 3, we consider graphs with girth at least $\ell - 1$. We prove that Conjecture 1.3(i) holds for this class of graphs, and observe that Conjecture 1.3(ii) also holds for such graphs. In Section 4, we present some concluding remarks.

2. Decomposition of regular graphs with prescribed girth

The main results of this section, Theorems 2.8 and 2.9, give sufficient conditions for regular graphs with prescribed girth to admit decompositions into paths of a given length. For that, we first apply Petersen's Factorization Theorem (Theorem 2.7) to obtain an initial trail decomposition, and then we improve the structure of this decomposition (see Lemma 2.1) to apply a variant of a result given in [3], which is called Disentangling Lemma (Lemma 2.4), obtaining the desired path decomposition.