ARTICLE IN PRESS

European Journal of Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Partitioning a triangle-free planar graph into a forest and a forest of bounded degree

François Dross [a], Mickael Montassier [a], Alexandre Pinlou [b]

[a] Université de Montpellier, LIRMM, CNRS, 161 rue Ada, 34095 Montpellier Cedex 5, France
[b] Université Paul-Valéry Montpellier 3, LIRMM, CNRS, 161 rue Ada, 34095 Montpellier Cedex 5, France

ARTICLE INFO

Article history:
Available online xxxx

ABSTRACT

An $(\mathcal{F}, \mathcal{F}_d)$-partition of a graph is a vertex-partition into two sets F and F_d such that the graph induced by F is a forest and the one induced by F_d is a forest with maximum degree at most d. We prove that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_5)$-partition. Moreover we show that if for some integer d there exists a triangle-free planar graph that does not admit an $(\mathcal{F}, \mathcal{F}_d)$-partition, then it is an NP-complete problem to decide whether a triangle-free planar graph admits such a partition.

1. Introduction

We only consider finite simple graphs, with neither loops nor multiple edges. Planar graphs we consider are supposed to be embedded in the plane. Consider i classes of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_i$. A $(\mathcal{G}_1, \ldots, \mathcal{G}_i)$-partition of a graph G is a vertex-partition into i sets $V_1, \ldots, V_i$ such that, for all $1 \leq j \leq i$, the graph $G[V_j]$ induced by V_j belongs to $\mathcal{G}_j$. In the following we will consider the following classes of graphs:

- $\mathcal{F}$ the class of forests,
- $\mathcal{F}_d$ the class of forests with maximum degree at most d,
- $\mathcal{D}_d$ the class of d-degenerate graphs (recall that a *d-degenerate graph* is a graph such that all subgraphs have a vertex of degree at most d),
- Δ_d the class of graphs with maximum degree at most d,
- $\mathcal{I}$ the class of empty graphs (i.e. graphs with no edges).

E-mail addresses: francois.dross@lirmm.fr (F. Dross), mickael.montassier@lirmm.fr (M. Montassier), alexandre.pinlou@lirmm.fr (A. Pinlou).

http://dx.doi.org/10.1016/j.ejc.2017.06.014

Table 1
Known results.

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$	The Four Color Theorem [1,2]
	$(\mathcal{I}, \mathcal{F}, \mathcal{F})$	Borodin [3]
	$(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$	Poh [8]
	$(\mathcal{F}, \mathcal{D}_2)$	Thomassen [10]
	$(\mathcal{I}, \mathcal{D}_3)$	Thomassen [11]
Planar graphs with girth 4	$(\mathcal{I}, \mathcal{I}, \mathcal{I})$	Grötzsch [6]
	$(\mathcal{F}, \mathcal{F})$	Folklore
	$(\mathcal{F}_5, \mathcal{F})$	Present paper (Theorem 3)
	$(\mathcal{I}, \mathcal{F})$	Open question (Question 1)
Planar graphs with girth 5	$(\mathcal{I}, \mathcal{F})$	Borodin and Glebov [4]

For example, an $(\mathcal{I}, \mathcal{F}, \mathcal{D}_2)$-partition of G is a vertex-partition into three sets V_1, V_2, V_3 such that $G[V_1]$ is an empty graph, $G[V_2]$ is a forest, and $G[V_3]$ is a 2-degenerate graph.

The Four Colour Theorem [1,2] states that every planar graph G admits a proper 4-colouring, that is G can be partitioned into four empty graphs, i.e. G has an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition. Borodin [3] proved that every planar graph admits an acyclic colouring with at most five colours (an acyclic colouring is a proper colouring in which every two colour classes induce a forest). This implies that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$-partition. Poh [8] proved that every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition. Thomassen proved that every planar graph admits an $(\mathcal{F}, \mathcal{D}_2)$-partition [10], and an $(\mathcal{I}, \mathcal{D}_3)$-partition [11]. However, there are planar graphs that do not admit any $(\mathcal{F}, \mathcal{F})$-partition [5]. Borodin and Glebov [4] proved that every planar graph of girth at least 5 (that is every planar graph with no triangles nor cycles of length 4) admits an $(\mathcal{I}, \mathcal{F})$-partition.

We focus on triangle-free planar graphs. Raspaud and Wang [9] proved that every planar graph with no triangles at distance at most 2 (and thus in particular every triangle-free planar graph) admits an $(\mathcal{F}, \mathcal{F})$-partition. However, it is not known whether every triangle-free planar graph admits an $(\mathcal{I}, \mathcal{F})$-partition. We pose the following questions:

Question 1. *Does every triangle-free planar graph admit an $(\mathcal{I}, \mathcal{F})$-partition?*

Question 2. *More generally, what is the lowest d such that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition?*

Note that proving $d = 0$ in Question 2 would prove Question 1. The main result of this paper is the following:

Theorem 3. *Every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_5)$-partition.*

This implies that $d \leq 5$ in Question 2. Our proof uses the discharging method. It is constructive and immediately yields an algorithm for finding an $(\mathcal{F}, \mathcal{F}_5)$-partition of a triangle-free planar graph in quadratic time.

Note that Montassier and Ochem [7] proved that not every triangle-free planar graph can be partitioned into two graphs of bounded degree (which shows that our result is tight in some sense).

Finally, we show that if for some d, there exists a triangle-free planar graph that does not admit an $(\mathcal{F}, \mathcal{F}_d)$-partition, then deciding whether a triangle-free planar graph admits such a partition is NP-complete. That is, if the answer to Question 2 is some $k > 0$, then for all $0 \leq d < k$, deciding whether a triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition is NP-complete. We prove this by reduction to Planar 3-Sat.

All presented results on vertex-partition of planar graphs are summarized in Table 1.

Theorem 3 will be proved in Section 2. Section 3 is devoted to complexity results.

Please cite this article in press as: F. Dross, et al., Partitioning a triangle-free planar graph into a forest and a forest of bounded degree, European Journal of Combinatorics (2017), http://dx.doi.org/10.1016/j.ejc.2017.06.014.