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On the generalised colouring numbers of graphs that exclude a fixed minor

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ABSTRACT

The generalised colouring numbers $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ were introduced by Kierstead and Yang as a generalisation of the usual colouring number, and have since then found important theoretical and algorithmic applications.

In this paper, we dramatically improve upon the known upper bounds for generalised colouring numbers for graphs excluding a fixed minor, from the exponential bounds of Grohe et al. to a linear bound for the r-colouring number col_r and a polynomial bound for the weak r-colouring number wcol_r. In particular, we show that if G excludes K_t as a minor, for some fixed $t \geq 4$, then $\mathrm{col}_r(G) \leq \binom{t-1}{2}(2r+1)$ and $\mathrm{wcol}_r(G) \leq \binom{r+t-2}{t-2}(t-3)(2r+1) \in \mathcal{O}(r^{t-1})$.

In the case of graphs G of bounded genus g, we improve the bounds to $\mathrm{col}_r(G) \leq (2g+3)(2r+1)$ (and even $\mathrm{col}_r(G) \leq 5r+1$ if $g = 0$, i.e. if G is planar) and $\mathrm{wcol}_r(G) \leq \left(2g + \binom{r+2}{2}\right)(2r+1)$.

1. Introduction

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there is a strict linear order $<_L$ of the vertices of G for which each vertex v has *back-degree* at most $k-1$, i.e. at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \mathrm{col}(G)$.

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Some generalisations of the colouring number of a graph have been studied in the literature. These include the *arrangeability* [4] used in the study of Ramsey numbers of graphs, the *admissibility* [15], and the *rank* [14] used in the study of the game chromatic number of graphs. But maybe the most natural generalisation of the colouring number is the two series col_r and wcol_r of *generalised colouring numbers* introduced by Kierstead and Yang [16] in the context of colouring games and marking games on graphs. As proved by Zhu [26], these invariants are strongly related to low tree-depth decompositions [18], and can be used to characterise bounded expansion classes of graphs (introduced in [20]) and nowhere dense classes of graphs (introduced in [21]). For more details on this connection, we refer the interested reader to [22].

The invariants col_r and wcol_r are defined in a way similar to the usual definition of the colouring number: the *r-colouring number* $\mathrm{col}_r(G)$ of a graph G is the minimum integer k such that there is a linear order $<_L$ of the vertices for which each vertex v can reach at most $k-1$ other vertices smaller than v (in the order $<_L$) with a path of length at most r, all internal vertices of which are greater than v. For the *weak r-colouring number* $\mathrm{wcol}_r(G)$, we do not require that the internal vertices are greater than v, but only that they are greater than the final vertex of the path. (Formal definitions will be given in Section 2.) As noticed already in [16], the two types of generalised colouring numbers are related by the inequalities

$$\mathrm{col}_r(G) \le \mathrm{wcol}_r(G) \le (\mathrm{col}_r(G))^r.$$

If we allow paths of any length (but still restrictions on the position of the internal vertices), we get the *∞-colouring number* $\mathrm{col}_\infty(G)$ and the *weak ∞-colouring number* $\mathrm{wcol}_\infty(G)$.

Generalised colouring numbers are an important tool in the context of algorithmic sparse graphs theory. They play a key role for example in the model-checking and enumeration algorithms for first-order logic on bounded expansion and nowhere dense graph classes [8,13,11], in Dvořák's linear time approximation algorithm for minimum distance-r dominating sets [7], and in the kernelisation algorithms for distance-r dominating sets [6,9].

An interesting aspect of generalised colouring numbers is that these invariants can also be seen as gradations between the colouring number $\mathrm{col}(G)$ and two important minor monotone invariants, namely the *tree-width* $\mathrm{tw}(G)$ and the *tree-depth* $\mathrm{td}(G)$ (which is the minimum height of a depth-first search tree for a supergraph of G [18]). More explicitly, for every graph G we have the following relations.

Proposition 1.1. (a) $\mathrm{col}(G) = \mathrm{col}_1(G) \le \mathrm{col}_2(G) \le \cdots \le \mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$;
(b) $\mathrm{col}(G) = \mathrm{wcol}_1(G) \le \mathrm{wcol}_2(G) \le \cdots \le \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$.

The equality $\mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$ was first proved in [10]; for completeness we include the proof in Section 2.2. The equality $\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ is proved in [22, Lemma 6.5].

As tree-width [12] is a fundamental graph invariant with many applications in graph structure theory, most prominently in Robertson and Seymour's theory of graphs with forbidden minors [24], it is no wonder that the study of generalised colouring numbers might be of special interest in the context of proper minor closed classes of graphs. As we shall see, excluding a minor indeed allows us to prove strong upper bounds for the generalised colouring numbers.

Using probabilistic arguments, Zhu [26] was the first to give a non-trivial bound for $\mathrm{col}_r(G)$ in terms of the densities of shallow minors of G. For a graph G excluding a complete graph K_t as a minor, Zhu's bound gives

$$\mathrm{col}_r(G) \le 1 + q_r,$$

where q_1 is the maximum average degree of a minor of G, and q_i is inductively defined by $q_{i+1} = q_1 \cdot q_i^{2i^2}$.

Grohe et al. [10] improved Zhu's bounds as follows:

$$\mathrm{col}_r(G) \le (crt)^r,$$

for some (small) constant c depending on t.

Our main results is an improvement of those bounds for the generalised colouring numbers of graphs excluding a minor.

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