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Limits of mappings

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ABSTRACT

In this paper we consider a simple algebraic structure — sets with a single endofunction. We shall see that from the point of view of structural limits, even this simplest case is both interesting and difficult. Nevertheless we obtain the shape of limit objects in the full generality, and we prove the inverse theorem in the case of quantifier-free limits.

1. Introduction

The aim of this paper is to construct analytic limit objects for convergent sequences of finite mappings $f_n : F_n \to F_n$ ("finite" meaning that the sets F_n are finite) and, conversely, to approximate a limit object by a finite mapping. This work originated within the scope of the recent studies of graph limits [18], and more precisely within the framework of structural limits [24]. As this framework is closely related to finite model theory, instead of describing mappings as $f : F \to F$ we shall define mappings as structures $\mathbf{F}$ (boldface) with signature $\{f\}$, where f is a unary function symbol, with domain F (same symbol as $\mathbf{F}$ but not boldface) and with interpretation of f denoted by $f_{\mathbf{F}}$. Hence $f_{\mathbf{F}} : F \to F$ and, for $u, v \in F$ we have the two following possible writings for the property that v is the image of u: either $f_{\mathbf{F}}(u) = v$ or $\mathbf{F} \models (f(u) = v)$.

In the general framework introduced in [24], the notion of convergence of structures is conceptualized by means of the convergence of the satisfaction probability of formulas in a fixed fragment of first-order logic: A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite structures is X-*convergent* (where X is a given fragment

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of first-order logic) if, for every formula $\phi \in X$ the probability $\langle \phi, \mathbf{A}_n \rangle$ of satisfaction of ϕ in $\mathbf{A}_n$ for a random (independent uniform) assignment of elements of the domain A_n of $\mathbf{A}_n$ to the free variables of ϕ converges as n grows to infinity. Three fragments of first-order logic, defining a gradation of three notions of convergence, will be of special interest: the fragment QF of quantifier-free formulas, the fragment $\mathrm{FO}^{\mathrm{local}}$ of *local* formulas (that is of formulas whose satisfaction only depends on a fixed neighborhood of the free variables) and the full FO fragment of all first-order formulas.

This framework allows one to consider limits of general combinatorial structures, and was applied to the study of limits of sparse graphs [11,22,25,26], matroids [17], and tree semi-lattices [8]. It is sometimes possible (although this is not the case in general [26]) to represent the limit by a particularly nice analytic object, called *modeling*, which is a structure whose domain is a standard Borel space endowed with a Borel probability measure, with the property that every (first-order) definable set is Borel measurable.

In order to make the motivation of this paper clear, we take time in Section 3 for a quick review of some of the fundamental notions and problems encountered in the domain of graph limits, and how they are related to the study of limits and approximations of algebras (that is of functional structures).

The first main result of this paper is the construction, for every FO-convergent sequence of finite mappings, of a modeling representing the FO-limit of the sequence (Theorem 1). As every sequence of finite structures contains an FO-convergent subsequence [24], this modeling can be used to represent the limits for the (weaker notions of) $\mathrm{FO}^{\mathrm{local}}$-convergence and QF-convergence. Theorem 1 is proved as a combination of general results about limit distributions (stated below as Theorem 3, see [24]) and methods developed in [25] for the construction of modeling limits of trees. As a consequence we are able to deduce the form of limits of mappings.

We shall also be interested in the inverse problems, which aim to determine which objects are X-limits of finite mappings (for a given fragment X of first-order logic). It should be noticed that although the inverse problem for QF-limits of graphs or hypergraphs has been completely solved, the inverse problem for $\mathrm{FO}^{\mathrm{local}}$-limits of graphs with uniformly bounded degrees (or equivalently for QF-limits of algebras with d involutions, see Section 3), which is known as Aldous–Lyons conjecture [4] is wide open.

In our (restricted) setting of algebras with a single function symbol, we solve the inverse problem for QF-limits. (Note that solving the inverse problem for QF-limits of algebras with 2 function symbols would imply solving the Aldous–Lyons conjecture.) The solution of the inverse problems for $\mathrm{FO}^{\mathrm{local}}$-limits and FO-limits of mappings, stated as Theorems 5 and 6 in Section 6 will be proved in a forthcoming paper.

2. Definitions and notations

Recall that a σ-structure $\mathbf{A}$ is defined by its *domain* A, its *signature* σ (which is a set of symbols of relations and functions together with their arities), and the interpretation all the relations and functions in σ as relations and functions on A.

The structures we consider here are structures with signature σ consisting of a single functional symbol f and (possibly) some unary symbols $M_1, \ldots, M_c$ (interpreted as a coloring). We call such structures *colored mappings* (or simply *mappings*).

Let $\mathbf{F}$ be such a mapping (with domain F). Then $f_{\mathbf{F}}$ is the interpretation of the symbol f in $\mathbf{F}$ (thus $f_{\mathbf{F}} : F \to F$). For a first-order formula ϕ with p free variables and a mapping $\mathbf{F}$ we define

$$\phi(\mathbf{F}) = \{(v_1, \ldots, v_p) \in F^p : \mathbf{F} \models \phi(v_1, \ldots, v_p)\}.$$

If $\mathbf{F}$ is finite (meaning that F is finite) we further define

$$\langle \phi, \mathbf{F} \rangle = \frac{|\phi(\mathbf{F})|}{|F|^p}.$$

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