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# A flow theory for the dichromatic number

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ABSTRACT

We transfer Tutte's theory for analyzing the chromatic number of a graph using nowhere-zero-coflows and -flows (NZ-flows) to the dichromatic number of a digraph and define Neumann-Lara-flows (NL-flows). We prove that any digraph whose underlying (multi-)graph is 3-edge-connected admits a NL-3-flow, and even a NL-2-flow in case the underlying graph is 4-edge connected. We conjecture that 3-edge-connectivity already guarantees the existence of a NL-2-flow, which, if true, would imply the 2-Color-Conjecture for planar graphs due to Víctor Neumann-Lara. Finally we present an extension of the theory to oriented matroids.



## 1. Introduction

Víctor Neumann-Lara [12] introduced the dichromatic number $\vec{\chi}(D)$ of a digraph $D$ as the smallest integer $k$ such that the vertices $V$ of $D$ can be colored with $k$ colors and each color class induces a directed acyclic graph. This is a proper generalization of the chromatic number: if $D$ is a digraph, with underlying graph $G$, where for every arc there exists an antiparallel one, then clearly $\chi(G) = \vec{\chi}(D)$. The purpose of this paper is to give a characterization of the dichromatic number in terms of coflows of the digraph and to develop a flow theory dual to this.

Neumann-Lara conjectured that the dichromatic number of an orientation of a simple planar graph is bounded by 2.

**Conjecture 1** (*Neumann-Lara [13]*)**.** *If $D$ is the orientation of a simple planar graph, then $\vec{\chi}(D) \le 2$.*

Using planar duality this 2-Color-Conjecture (2CC) is equivalent to a bound of 2 on a certain flow, which we will call NL-flow. The non-existence of antiparallel arcs in the planar graph dualizes to 3-edge-connectivity of the underlying (multi-)graph. We will show that, in contrast to the classical

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case of 4-coloring of planar graphs where the Petersen graph is an obstruction to the existence of a NZ-4-flow [17], any orientation of the Petersen graph admits a NL-2-flow. If one could show that a NL-2-flow exists in any 3-edge-connected-digraph, this would verify the 2-Color-Conjecture.

Using techniques from [10] we can show that 4-edge connectivity of the underlying graph is sufficient for the existence of a NL-2-flow, while we can only prove the existence of a NL-4-flow in the 3-edge-connected case. Applying a result of Seymour [15] Matt DeVos [3] helped us to improve this and to verify the existence of a NL-3-flow in the 3-edge-connected case.

Our notation is fairly standard and, if not explicitly defined, should follow the books of Diestel [4] for graphs and Björner et. al. [1] for oriented matroids. Note that all our digraphs may have parallel and antiparallel arcs. A *weak cycle* in a digraph $D$ is a not necessarily directed cycle.

## 2. Neumann-Lara-flows and -coflows

We first recall Tutte's definition of a *NZ-k-flow*. Let $G = (V, E)$ be a bridgeless graph. A *nowhere-zero-k-flow* $(D, \phi)$, or a *NZ-k-flow* for short is an orientation $D = (V, A)$ of $G$ together with a map $\phi : A \to \{\pm 1, \dots, \pm(k-1)\}$ satisfying Kirchhoff's law of flow conservation

$$\sum_{a\in\delta^-(v)} \phi(a) = \sum_{a\in\delta^+(v)} \phi(a) \quad \text{for all } v \in V.$$

We generalize this to digraphs as follows: Let $D = (V, A)$ be a directed graph. A mapping $f = (f_1, f_2) : A \to \mathbb{Z}^2$ is called a Neumann-Lara-*flow* or *NL-flow* for short, if both components of $f$ satisfy Kirchhoff's law of flow conservation and, whenever $f_1(a) = 0$, for some arc $a \in A$, then necessarily $f_2(a) > 0$.

A NL-flow is a *NL-k-flow*, if additionally

$$|f_1(a)| < k \quad \text{for all } a \in A.$$

Note that the existence of a NL-$k$-flow may depend on the orientation of the digraph $D$, e.g. a connected digraph $D$ has a NL-1-flow if and only if it is strongly connected.

A mapping $f^* = (f_1^*, f_2^*) : A \to \mathbb{Z}^2$ is a Neumann-Lara-*coflow*, a *NL-coflow* for short, if it satisfies Kirchhoff's law for cycles, i.e. for each weak cycle $C$ of $D$

$$\sum_{a\in C^+} f_i^*(a) = \sum_{a\in C^-} f_i^*(a), \tag{1}$$

where $C^+$ and $C^-$ denote the arcs of $C$ that are traversed in forward resp. backward direction and, whenever $f_1^*(a) = 0$, for some arc $a \in A$, then necessarily $f_2^*(a) > 0$.

A NL-coflow is a *NL-k-coflow*, if in addition

$$|f_1^*(a)| < k \quad \text{for all } a \in A.$$

The following theorem was our motivation to define and study NL-flows.

**Theorem 1.** *Let $D = (V, A)$ be a loopless directed graph. Then $D$ has a NL-k-coflow if and only if it has dichromatic number at most k.*

**Proof.** Clearly, we may assume that $D$ is connected. Let $f^* : A \to \mathbb{Z}^2$ be a NL-$k$-coflow. We define a coloring of $c$ using the colors $\{0, 1, \dots, k-1\}$ as follows. Choose an arbitrary vertex $v \in V$ which receives color zero $c(v) = 0$. Now let $w$ be another vertex and $P_1$ be a (not necessarily directed) $v$–$w$-path in $D$. Then we define the preliminary color $\tilde{c}(w)$ of $w$ as

$$\tilde{c}(w) = \sum_{a\in P_1^+} f_1^*(a) - \sum_{a\in P_1^-} f_1^*(a),$$

where $P_1^+$ and $P_1^-$ denote the arcs of $P_1$ that are traversed in positive resp. negative direction and claim that this value is independent of the chosen path. Namely, if $P_2$ is another such path, then the

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