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On the minimum number of edges in triangle-free 5-critical graphs

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ABSTRACT

Kostochka and Yancey proved that every 5-critical graph G satisfies: $|E(G)| \geq \frac{9}{4}|V(G)| - \frac{5}{4}$. A construction of Ore gives an infinite family of graphs meeting this bound.

We prove that there exists $\epsilon, \delta > 0$ such that if G is a 5-critical graph, then $|E(G)| \geq (\frac{9}{4} + \epsilon)|V(G)| - \frac{5}{4} - \delta T(G)$ where $T(G)$ is the maximum number of vertex-disjoint cliques of size three or four where cliques of size four have twice the weight of a clique of size three. As a corollary, a triangle-free 5-critical graph G satisfies: $|E(G)| \geq (\frac{9}{4} + \epsilon)|V(G)| - \frac{5}{4}$.

1. Introduction

A graph G is *k-critical* if G is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. Since the minimum degree of k-critical graph is at least $k-1$, it follows trivially that the number of edges in a k-critical graph G is at least $\frac{k-1}{2}|V(G)|$. In a recent landmark paper, Kostochka and Yancey [7] improved this lower bound as follows.

Theorem 1.1 (*Kostochka, Yancey [8]*)**.** *If G is a k-critical graph, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1}\right)|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

Theorem 1.1 is tight for all k. In particular, K_k satisfies the formula with equality, yet there also exist an infinite family of k-critical graphs matching this bound. Theorem 1.1 confirmed a conjecture of Gallai [1] on the minimum asymptotic ratio of edges to vertices in a k-critical graph and almost

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proved a conjecture of Ore [11] on the exact lower bound for the number of edges in a k-critical graph on a fixed number of vertices. It is natural to wonder whether the lower bound may be improved by restricting to a subclass of k-critical graphs. In particular, would excluding certain subgraphs increase the asymptotic ratio of edges to vertices in a k-critical graph?

For $k = 4$, Theorem 1.1 states that a 4-critical graph G on n vertices has at least $\frac{5n-2}{3}$ edges. A construction of Thomas and Walls [13] yields an infinite family of 4-critical triangle-free graphs whose asymptotic ratio of edges to vertices is also $5/3$. In another paper [12], the author proved the following theorem.

Theorem 1.2. *There exists $\epsilon > 0$ such that if G is a 4-critical graph of girth at least five, then*

$$|E(G)| \geq \left(\frac{5}{3} + \epsilon\right) |V(G)| - \frac{2}{3}.$$

This suggests that excluding certain subgraphs – for $k = 4$, the triangle and 4-cycle – can improve the lower bound. What then is a natural set of subgraphs to exclude for general k? The construction of Thomas and Walls can be extended to yield an infinite family of k-critical K_{k-1}-free graphs whose asymptotic ratio of edges to vertices is $\frac{k}{2} - \frac{1}{k-1}$, the same ratio as in Theorem 1.1.

What if we exclude even smaller cliques? In a work predating Kostochka and Yancey, Krivelevich [9] investigated this question, though the bounds he proved are worse than those of Kostochka and Yancey. Kostochka and Stiebitz [5] proved that ratio of edges to vertices in a k-critical K_s-free graph is at least $k - o(k)$ when s is fixed; this is a multiplicative factor of 2 improvement and is best possible. For details about these and other related results as well as more history about our problem in general, see the extensive survey of Kostochka [4], in particular Section 5. Also in that section of the survey, Kostochka says that finding the average degree of triangle-free k-critical graphs for small and moderate k is an interesting open problem.

On the other extreme, instead of excluding fixed sized cliques, it is natural to wonder what is the largest clique that can be excluded such that the tight bound of Kostochka and Yancey can be improved. As noted above, K_{k-1} does not suffice. We make the following conjecture that excluding K_{k-2} does indeed suffice.

Conjecture 1.3. *For every $k \geq 4$, there exists $\epsilon_k > 0$ such that if G is a k-critical K_{k-2}-free graph, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \epsilon_k\right) |V(G)| - \frac{k(k-3)}{2(k-1)}.$$

The conjecture is vacuously true for $k = 4$ since there does not exist a 4-critical K_2-free graph. Hence, Theorem 1.2 can be viewed as the appropriate analogue for Conjecture 1.3 with $k = 4$ where K_2-free is replaced by $\{C_4, K_3\}$-free. The subject of this paper is to consider the case when $k = 5$. In fact, we prove Conjecture 1.3 for $k = 5$ as follows.

Theorem 1.4. *There exists $\epsilon > 0$ such that if G is a 5-critical triangle-free graph, then*

$$|E(G)| \geq \left(\frac{9}{4} + \epsilon\right) |V(G)| - \frac{5}{4}.$$

In fact, our main theorem shows that $\epsilon = \frac{1}{84}$ suffices in Theorem 1.4. We should also mention that Conjecture 1.3 has recently been proven for $k = 6$ [2] and also for large k [10]. Strangely, the range of moderate numbers starting with $k = 7$ remains open; indeed, these seem to be the hardest cases. Before we discuss how Theorem 1.4 is proved, we need to discuss the family of graphs which attain the bound in Theorem 1.1 since these are central to the proof of Theorem 1.4.

Definition 1.5. An *Ore-composition* of graphs G_1 and G_2 is a graph obtained by the following procedure:

1. delete an edge xy from G_1;
2. split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree;
3. identify x with z_1 and identify y with z_2.