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Counting configuration-free sets in groups

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ABSTRACT

We provide asymptotic counting for the number of subsets of given size which are free of certain configurations in finite groups. Applications include sets without solutions to equations in non-abelian groups, and linear configurations in abelian groups defined from group homomorphisms. The results are obtained by combining the methodology of hypergraph containers joint with arithmetic removal lemmas. Random sparse versions and threshold probabilities for existence of configurations in sets of given density are presented as well.

1. Introduction

The study of sparse (and probabilistic) analogues of results in extremal combinatorics has become a very active area of research in extremal and random combinatorics (see e.g. the survey by Conlon [7]). One starting point is *Szemerédi Theorem* [39] on the existence of arbitrarily long arithmetic progressions in sets of integers with positive upper density. This seminal result and the tools arising in its many proofs have been enormously influential in the development of modern discrete mathematics. Nowadays a large proportion of the research in additive combinatorics is inspired by these achievements.

Sparse analogues of Szemerédi Theorem started in Kohayakawa, Rödl and Łuczack [18] by studying the threshold probability for a random set of the integer interval $[1, n]$ whose subsets of given density contain asymptotically almost surely (a.a.s.) 3-term arithmetic progressions. The extension of the result to k-term arithmetic progressions was a breakthrough obtained independently, and by different methods, by Conlon and Gowers [8] and by Schacht [31]. There is still another more recent proof based on combinatorial arguments due to Saxton, and Thomason [30] and by Balogh, Morris, and

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Samotij [3]. The approach in the above two papers is based on a methodology building on the structure of independent sets in hypergraphs. *Hypergraphs containers* (as it is named in [30]) provide a general framework to attack a wide variety of problems which can be encoded by uniform hypergraphs. The philosophy behind this method is that, for a large class of uniform hypergraphs which satisfy mild conditions, one can find a small collection of sets of vertices (which are called *containers*) which contain all independent sets of the given hypergraph, thus providing sensible upper bounds on the number of independent sets.

In addition to important applications in combinatorics, the two works above mentioned also contain arithmetic applications, providing in particular a new proof of the sparse Szemerédi Theorem. One important ingredient of these proofs, explicitly exposed in [3], is the so-called *Varnavides Theorem* [41]. This is the robust counterpart of Szemerédi Theorem: once a set has positive density, it does not only have one but a positive proportion of the total number of k-term arithmetic progressions. This phenomenon is the number theoretical counterpart of the supersaturation phenomenon in the graph setting.

Nowadays there is a rich theory dealing with these type of results, which are rephrased under the name of *Arithmetic Removal Lemmas*. The idea behind them can be traced back to the proof of Roth's Theorem by Ruzsa and Szemerédi [28] and was first formulated by Green [17] for a linear equation in an abelian group by using methods of Fourier analysis. The picture was complemented independently by Shapira [34] and by Král', Serra and Vena [21] by proving a removal lemma for linear systems in the integers. These results have been extended in several directions, including arithmetic removal lemmas for a single equation in non-abelian groups, for linear systems over finite fields and for integer linear systems over finite abelian groups (see [20–22,34]).

These extensions of Green's Arithmetic Removal Lemma provide proofs of the Szemerédi Theorem in general abelian groups (see also [38]), but cannot handle the robust versions of the multidimensional Szemerédi Theorem (see for instance [36] on Furstenberg and Katznelson work [15]) or, more generally, the appearance and enumeration of finite configurations in dense subsets in abelian groups (as seen in Tao [40, Theorem B.1]). As a consequence, the above mentioned arithmetic removal lemmas cannot be used to show the sparse counterparts of these results (see [3,8,31]).

The main contribution of this paper is to combine the method of hypergraph containers with a removal lemma for group homomorphisms due to Vena [42], which unifies and extends previous results concerning arithmetic removal lemmas. This combination provides a more general and flexible counting result, which allows for new applications which are developed in the paper. We next summarize the contents of the paper and its contributions.

Section 2 contains a version of the hypergraph container method in the more general framework of configuration systems. A system of configurations of degree k on a ground set G is a pair (S, G) where $S \subset G^k$ is a subset of k-tuples of G. Our goal is to have a tool to bound the number of S-free subsets of k-tuples from G. We introduce some appropriate parameters on configuration systems and reformulate in Theorem 2.4 the counting result of [3] in this more general language, which will be needed for the applications developed later on in the paper. We define a class of systems of configurations, which we call *normal*, which satisfy some natural properties shared by all the configurations we meet in the arithmetic applications.

Section 3 extends to systems of configurations the study of the threshold probability for the stability in random subsets. Given a system (S, G) of configurations, a subset $X \subset G$ is (δ, S)-stable if, for every subset $X' \subset X$ with density at least δ in X, the set of k-tuples $(X')^k$ intersects S (X' contains a configuration form S.) We give the threshold probabilities for this notion of stability in random subsets of G in Theorems 3.2 and 3.5 for normal systems of configurations. The transition probability for the stability, or 1-statement, can be deduced in a simple way from existing results. The transition probability for nonstability, or 0-statement, follows by carefully applying standard techniques, but it is more technically involved. Since stability is not a monotone property, the two transition probabilities may not coincide, and we provide a necessary condition on configurations systems to ensure that there is a threshold transition.

Section 4 discusses systems of configurations defined by kernels of homomorphisms in abelian groups. In this case the ground set G is an abelian group and S is the kernel of a group homomorphism $M : G^k \to G^m$. This extends in a natural way the study of solutions of linear systems in abelian

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