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Inductive tools for connected delta-matroids and multimatroids

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A B S T R A C T

We prove a splitter theorem for tight multimatroids, generalizing the corresponding result for matroids, obtained independently by Brylawski and Seymour. Further corollaries give splitter theorems for delta-matroids and ribbon graphs.

1. Introduction

A *matroid* $M = (E, \mathcal{B})$ is a finite *ground set* E together with a non-empty collection of subsets of the ground set, $\mathcal{B}$, that are called *bases*, satisfying the following conditions, which are stated in a slightly different way from what is most common in order to emphasize the connection with other combinatorial structures discussed in this paper.

1. If B_1 and B_2 are bases and $x \in B_1 \triangle B_2$, then there exists $y \in B_1 \triangle B_2$ such that $B_1 \triangle \{x, y\}$ is a basis.
2. All bases are equicardinal.

Matroid theory is often thought of as a generalization of graph theory, as a matroid $(M, \mathcal{B})$ may be constructed from a graph G by taking E to be set of edges of G and $\mathcal{B}$ to be the edge sets of maximal spanning forests of G. Graph theory and matroid theory are mutually enriching: many results in graph theory have been generalized to matroids, and results in matroid theory have sometimes been proved before the corresponding specialization in graph theory. In [13], Chun, Moffatt, Noble

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and Rueckriemen showed that the mutually-enriching relationship between graphs and matroids is analogous to the mutually-enriching relationship between cellularly-embedded graphs, which we view as ribbon graphs, and objects called *delta-matroids*. They gave further evidence for this by establishing several new results for delta-matroids in [12], each of which was inspired by a previously known result concerning ribbon graphs.

Delta-matroids were extensively studied by Bouchet in the 1980s, but until recently had been little studied since that foundational work. In addition to [12,13], where the authors were led to delta-matroids by studying ribbon graphs, they have been studied extensively by Brijder and Hoogeboom who were originally interested in the principal pivot transform in binary matrices (see, for example, [7–9]).

A *delta-matroid* $D = (E, \mathcal{F})$ is a finite *ground set* E together with a non-empty collection of subsets of the ground set, $\mathcal{F}$, that are called *feasible sets*, satisfying the following condition known as the *symmetric exchange axiom*. If F_1 and F_2 are feasible sets and $x \in F_1 \triangle F_2$, then there exists $y \in F_1 \triangle F_2$ such that $F_1 \triangle \{x, y\}$ is a feasible set. Note that we allow $y = x$. It follows immediately from the definitions that every matroid is a delta-matroid. In fact, the axiom for the feasible sets of a delta-matroid corresponds exactly to (1) in the axioms we gave earlier for the bases of a matroid. A delta-matroid is said to be *even* if the sizes of its feasible sets all have the same parity. Thus a matroid is an even delta-matroid.

As in many other areas of mathematics, structural results on matroids often require an assumption of some level of connectivity of the matroid. In [15], Geelen defined connectivity for delta-matroids as follows. Given delta-matroids $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$ with disjoint ground sets, their *direct sum*, written $D_1 \oplus D_2$, is the delta-matroid with ground set $E_1 \cup E_2$ and collection of feasible sets $\{F_1 \cup F_2 : F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\}$. If $D = D_1 \oplus D_2$ then we say that $E(D_1)$ and $E(D_2)$ are *separators* of D. If X is a separator of a delta-matroid D and $\emptyset \neq X \neq E(D)$ then we say that X is a *proper separator* of D. A delta-matroid D is *disconnected* if it has a proper separator. Otherwise D is *connected*. Clearly the matroids that satisfy the definition of delta-matroid connectivity are exactly those that satisfy the well-known definition of matroid connectivity [19]. Moreover when applied to matroids, the definition of a separator in a delta-matroid is exactly the same as that of a separator in a matroid [19]. Our aim is to study the effect on connectivity of removing elements from a delta-matroid. As a consequence we provide useful tools for inductive proofs of results concerning 2-connected ribbon graphs, which we define later.

Deletion and contraction are the two natural ways in which to remove an element from a matroid or delta-matroid. For a delta-matroid $D = (E, \mathcal{F})$, and $e \in E$, if e is in every feasible set of D, then we say that e is a *coloop of* D. If e is in no feasible set of D, then we say that e is a *loop of* D. If e is not a coloop, then, following Bouchet and Duchamp [6], we define D *delete* e, written $D\backslash e$, to be

$$D\backslash e = (E - e, \{F : F \in \mathcal{F} \text{ and } F \subseteq E - e\}).$$

If e is not a loop, then we define D *contract* e, written D/e, to be

$$D/e = (E - e, \{F - e : F \in \mathcal{F} \text{ and } e \in F\}).$$

If e is a loop or coloop, then $D/e = D\backslash e$.

Both $D\backslash e$ and D/e are delta-matroids (see [6]). Let D' be a delta-matroid obtained from D by a sequence of deletions and contractions. Then D' is independent of the order of the deletions and contractions used in its construction (see [6]) and D' is called a *minor* of D. We let $D|A$ denote $D\backslash(E - A)$. All of these definitions are entirely consistent with the corresponding better-known definitions for matroids.

Two early results describing the effect of deleting or contracting an element from a matroid are the following. The first was proved by Tutte [21] and the second independently by Brylawski [10] and Seymour [20].

Theorem 1.1. *Let e be an element of a connected matroid M. Then either $M\backslash e$ or M/e is connected.*

Theorem 1.2. *Let N be a connected minor of a connected matroid M and let e be an element of $E(M) - E(N)$. Then either M/e or $M\backslash e$ is connected and has N as a minor.*