European Journal of Combinatorics 63 (2017) 124–133

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Multicolour Ramsey numbers of paths and even cycles

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ARTICLE INFO

Article history:
Received 16 September 2016
Accepted 1 March 2017
Available online 30 March 2017

ABSTRACT

We prove new upper bounds on the multicolour Ramsey numbers of paths and even cycles. It is well known that $(k-1)n + o(n) \leqslant R_k(P_n) \leqslant R_k(C_n) \leqslant kn + o(n)$. The upper bound was recently improved by Sárközy who showed that $R_k(C_n) \leqslant \left(k - \frac{k}{16k^3+1}\right) n + o(n)$. Here we show $R_k(C_n) \leqslant (k - \frac{1}{4})n + o(n)$, obtaining the first improvement to the coefficient of the linear term by an absolute constant.

1. Introduction

Ramsey theory is one of the central areas of study in combinatorics and a key problem in the field is that of determining the Ramsey numbers of graphs, defined as follows. For a graph G, the *Ramsey number* $R_k(G)$ is the least integer N such that any colouring of the edges of the complete graph K_N on N vertices with k colours yields a monochromatic copy of G. The existence of Ramsey numbers is guaranteed by Ramsey's classical result [15], but in the case $k \geqslant 3$, determining the value of $R_k(G)$ for a given graph G is in most cases difficult. There are only a few graphs G for which we know $R_k(G)$ exactly and often one has to settle for bounds on this quantity. In this paper we focus on the case where G is the n-vertex path P_n, and the case where n is even and G is the n-vertex cycle C_n. The two-colour Ramsey number of a path was completely determined by Gerencsér and Gyárfás [9] who showed that for $n \geqslant 2$

$$R_2(P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor .$$

For three colours, Faudree and Schelp [6] conjectured that

$$R_3(P_n) = \begin{cases} 2n-2 & \text{for } n \text{ even}, \\ 2n-1 & \text{for } n \text{ odd}. \end{cases}$$

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http://dx.doi.org/10.1016/j.ejc.2017.03.002
0195-6698/

This conjecture was resolved for large n by Gyárfás, Ruszinkó, Sárközy and Szemerédi [10] but for $k \geqslant 4$ much less is known. A well-known upper bound $R_k(P_n) \leqslant kn$ follows easily by observing that any k-colouring of the complete graph on kn vertices contains a colour class with at least $(kn-1)\frac{n}{2}$ edges by the pigeonhole principle. A result of Erdős and Gallai [4] (Lemma 2) then implies that any graph on kn vertices with this many edges contains a copy of P_n. Despite the simplicity of this observation, the bound was only recently improved upon by Sárközy [17] who proved a stability version of Lemma 2 and showed that for $k \geqslant 4$ and n sufficiently large,

$$R_k(P_n) \leqslant \left(k - \frac{k}{16k^3+1}\right) n \,.$$

In this paper we improve on the above result for all $k \geqslant 4$ reducing the upper bound on $R_k(P_n)$ by an amount that does not deteriorate as k grows. Our method is similar to that of [17] in that we also use results of Erdős and Gallai [4], and Kopylov [12] to bound the number of edges in the densest two colours. Our improvement comes from using more information about the densest colour in order to obtain stronger bounds on the number of edges in the second densest.

Theorem 1. *For $k \geqslant 4$ and all $n \geqslant 64k$,*

$$R_k(P_n) \leqslant \left(k - \frac{1}{4} + \frac{1}{2k}\right) n \,.$$

If n is much larger we can in fact slightly improve on this bound and extend it to even cycles, see Theorem 2.

Since P_n is a subgraph of C_n we have $R_k(P_n) \leqslant R_k(C_n)$. It is believed that for fixed k and even n the Ramsey numbers $R_k(P_n)$ and $R_k(C_n)$ are asymptotically equal. This is due to an application of the regularity lemma and the notion of connected matchings pioneered by Łuczak in [13]. Progress on these two problems therefore track each other closely. In the case of two colours Faudree and Schelp [5], and independently Rosta [16] showed that $R_2(C_n) = \frac{3n}{2}+1$ for even $n \geqslant 6$. For three colours, Benevides and Skokan [1] proved that $R_3(C_n) = 2n$ for sufficiently large even n. For $k \geqslant 4$ colours, again very little is known. Łuczak, Simonovits and Skokan [14] showed that for n even, $R_k(C_n) \leqslant kn + o(n)$, and recently Sárközy [17] improved this upper bound to $\left(k - \frac{k}{16k^3+1}\right) n + o(n)$. Here we obtain a strengthening of Theorem 1 for large n.

Theorem 2. *For $k \geqslant 4$ and n even*

$$R_k(C_n) \leqslant \left(k - \frac{1}{4}\right) n + o(n) \,.$$

It is interesting to note that odd cycles behave very differently in this context. Recently the second author and Skokan [11] showed, via analytic methods, that for $k \geqslant 4$ and n odd and sufficiently large, $R_k(C_n) = 2^{k-1}(n-1)+1$. This resolved a conjecture of Bondy and Erdős [3] for large n.

Let us now briefly discuss lower bounds. Constructions based on finite affine planes (see [2]) show that $R_k(P_n) \geqslant (k-1)(n-1)$, when $k-1$ is a prime power and this lower bound is thought to be closer to the truth than our upper bound. Yongqi, Yuansheng, Feng, and Bingxi [18] provide a construction which shows that $R_k(C_n) \geqslant (k-1)(n-2)+2$ for any k and for even n. This construction can easily be modified to give a lower bound on $R_k(P_n)$ for any k and any n. We sketch this construction below.

To see that $R_k(P_n) \geqslant 2(k-1)\left(\lfloor \frac{n}{2} \rfloor - 1\right)+1$, consider a complete graph G on vertices $\{0, 1, \ldots, 2k-3\}$ and for $1 \leqslant i \leqslant k-1$ colour the edges from vertex i to vertices $i+1, \ldots, i+k-2$ and the edges from vertex $i+k-1$ to vertices $i+k, \ldots, i+2k-3$ (taken modulo $2k-2$) with colour c_i. Then each colour $c_1, \ldots, c_{k-1}$ consists of two vertex-disjoint stars, each on $k-1$ vertices. The remaining edges are those of the form $\{j, j+k-1\}$ for $j = 0, \ldots, k-2$ which are coloured with the final colour c_k. The final colour forms a matching on $k-1$ edges. Construct G' by 'blowing up' each vertex i of G into a set V_i of $\lfloor \frac{n}{2} \rfloor - 1$ vertices and colour the edges within V_i with colour c_k. Edges between sets V_i and V_j in G' are coloured with the same colour as the edge $\{i, j\}$ in G.

There is no monochromatic P_n in G' because in colours $c_1, \ldots, c_{k-1}$, components are bipartite with smallest part size $\lfloor \frac{n}{2} \rfloor - 1$, hence cannot contain a P_n. The components in colour c_k have less than n