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Pentavalent symmetric graphs admitting vertex-transitive non-abelian simple groups

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ABSTRACT

A graph Γ is said to be *symmetric* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the arc set of Γ. Let G be a finite non-abelian simple group and let Γ be a connected pentavalent symmetric graph with $G \leq \mathrm{Aut}(\Gamma)$. In this paper, we show that if G is transitive on the vertex set of Γ, then either $G \trianglelefteq \mathrm{Aut}(\Gamma)$ or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \leq T$ and (G, T) is one of 58 possible pairs of non-abelian simple groups. In particular, if G is transitive on the arc set of Γ, then (G, T) is one of 17 possible pairs, and if G is regular on the vertex set of Γ, then (G, T) is one of 13 possible pairs, which improves the result on pentavalent symmetric Cayley graph given by Fang, et al. (2011).

1. Introduction

Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *regular* on Ω if for any two points there is a unique element of G mapping one to the other. Denote by $\mathbb{Z}_n$, D_n, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Throughout this paper, all groups and graphs are finite, and all graphs are simple and undirected. For a graph Γ, we denote its vertex set and automorphism group by $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, respectively. A graph Γ is said to be *G-vertex-transitive* for $G \leq \mathrm{Aut}(\Gamma)$ if G acts transitively on $V(\Gamma)$, *G-regular* if G acts regularly on $V(\Gamma)$, and *G-symmetric* if G acts transitively on the arc set of Γ (an arc is an ordered pair of adjacent vertices). In particular, Γ is *vertex-transitive* or *symmetric* if it is $\mathrm{Aut}(\Gamma)$-vertex-transitive or $\mathrm{Aut}(\Gamma)$-symmetric, respectively. The graph Γ is a *Cayley graph* on G if G is regular on the vertex set of Γ, and the Cayley graph is *normal* if G is a normal subgroup of $\mathrm{Aut}(\Gamma)$.

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Let G be a non-abelian simple group and let Γ be a pentavalent symmetric G-vertex-transitive graph. In this paper, we show that either $G \trianglelefteq \mathrm{Aut}(\Gamma)$ or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \le T$ and (G, T) is one of 58 possible pairs of non-abelian simple groups. The motivation of this investigation comes from the two extreme cases, that is, G-symmetric and G-regular.

There are two steps to study a symmetric graph Γ — the first step is to investigate the normal quotient graph for a normal subgroup of an arc-transitive group of automorphisms (see Section 2 for the definition of quotient graph), and the second step is to reconstruct the original graph Γ from the normal quotient by using covering techniques. This is usually done by taking a maximal normal subgroup such that the normal quotient graph has the same valency as the graph Γ, and in this case, the normal quotient graph is called a *basic graph* of Γ. The situation seems to be somewhat more promising with 2-arc-transitive graphs (a 2-arc is a directed path of length 2), and the strategy for the structural analysis of these graphs, based on taking normal quotients, was first laid out by Praeger (see [22,23]). The strategy works for locally primitive graphs, that is, vertex-transitive graphs with vertex stabilizers acting primitively on the corresponding neighbors sets (see [21,24]). For more results, refer to [9,18] for example. For a non-abelian simple group G, the G-symmetric graphs are important basic graphs and have received wide attention. For example, Fang and Praeger [6,7] classified G-symmetric graphs admitting G as Suzuki simple groups or Ree simple groups acting transitively on the set of 2-arcs of the graphs. For cubic G-symmetric graph, it was proved by Li [17] that either G is a normal in $\mathrm{Aut}(\Gamma)$, or $(G, \mathrm{Aut}(\Gamma)) = (A_7, A_8)$, (A_7, S_8), $(A_7, 2.A_8)$, (A_{15}, A_{16}) or $(\mathrm{GL}(4, 2), \mathrm{AGL}(4, 2))$. Fang et al. [3] proved that none of the above five pairs can happen, that is, G is always normal in $\mathrm{Aut}(\Gamma)$. In this paper, we show that if Γ be a connected pentavalent G-symmetric graph, then either $G \trianglelefteq \mathrm{Aut}(\Gamma)$ or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \le T$ and (G, T) is one of 17 possible pairs of non-abelian simple groups.

Investigation of Cayley graphs on a non-abelian simple group is currently a hot topic in algebraic graph theory. One of the most remarkable achievements is the complete classification of connected trivalent symmetric non-normal Cayley graphs on finite non-abelian simple groups. This work began in 1996 by Li [17], and he proved that a connected trivalent symmetric Cayley graph Γ on a non-abelian simple group G is either normal or $G = A_5, A_7, \mathrm{PSL}(2, 11), M_{11}, A_{11}, A_{15}, M_{23}, A_{23}$ or A_{47}. In 2005, Xu et al. [30] proved that either Γ is normal or $G = A_{47}$, and two years later, Xu et al. [31] further showed that if $G = A_{47}$ and Γ is not normal, then Γ must be 5-arc-transitive and up to isomorphism there are exactly two such graphs.

Let Γ be a connected symmetric Cayley graph Γ on a non-abelian simple group G. Fang, Praeger and Wang [8] developed a general method to investigate the automorphism group $\mathrm{Aut}(\Gamma)$, and using this, the symmetry of Cayley graphs of valency 4 or 5 has been investigated in [4,5]. Now let Γ be non-normal and of valency 5. Fang et al. [5] proved that if $\mathrm{Aut}(\Gamma)$ is quasiprimitive then $(G, \mathrm{soc}(\mathrm{Aut}(\Gamma))) = (A_{n-1}, A_n)$, where either $n = 60 \cdot k$ with $k \big| 2^{15} \cdot 3$ and $k \ne 3, 4, 6, 8$ or $n = 10 \cdot m$ with $m \big| 8$, and if $\mathrm{Aut}(\Gamma)$ is not quasiprimitive then there is a maximal intransitive normal subgroup K of $\mathrm{Aut}(\Gamma)$ such that the socle of $\mathrm{Aut}(\Gamma)/K$, denoted by $\bar{L}$, is a simple group containing $\bar{G} = GK/K \cong G$ properly, where $(\bar{G}, \bar{L}) = (\Omega^-(8, 2), \mathrm{PSp}(8, 2))$, (A_{14}, A_{16}), (A_{n-1}, A_n) with $n \ge 6$ such that n is a divisor of $2^{17} \cdot 3^2 \cdot 5$ or $\bar{G} = \bar{L}$ is isomorphic to an irreducible subgroup of $\mathrm{PSL}(d, 2)$ for $4 \le d \le 17$ and the 2-part $|G|_2 > 2^d$. In this paper, we prove that $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \le T$ and (G, T) is one of 13 possible pairs of non-abelian simple groups. The following is the main result of this paper.

Theorem 1.1. *Let G be a non-abelian simple group and let Γ be a connected pentavalent symmetric G-vertex-transitive graph. Then either $G \trianglelefteq \mathrm{Aut}(\Gamma)$, or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \le T$ and $(G, T) = (\Omega_8^-(2), \mathrm{PSp}(8, 2))$, (A_{14}, A_{16}), $(\mathrm{PSL}(2, 8), A_9)$ or (A_{n-1}, A_n) with $n \ge 6$ and $n \big| 2^9 \cdot 3^2 \cdot 5$.*

Corollary 1.2. *Let G be a non-abelian simple group and let Γ be a connected pentavalent G-symmetric graph. Then either $G \trianglelefteq \mathrm{Aut}(\Gamma)$, or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup T such that $G \le T$ and $(G, T) = (A_{n-1}, A_n)$ with $n = 2 \cdot 3, 2^2 \cdot 3, 2^4, 2^3 \cdot 3, 2^5, 2^2 \cdot 3^2, 2^4 \cdot 3, 2^3 \cdot 3^2, 2^5 \cdot 3, 2^4 \cdot 3^2, 2^6 \cdot 3, 2^5 \cdot 3^2, 2^7 \cdot 3, 2^6 \cdot 3^2, 2^7 \cdot 3^2, 2^8 \cdot 3^2$ or $2^9 \cdot 3^2$. Moreover, if $T = A_6$, then $\mathrm{Aut}(\Gamma) = S_6$ and if $T = A_{2^5}$, $A_{2^7 \cdot 3}$, $A_{2^7 \cdot 3^2}$, $A_{2^8 \cdot 3^2}$ or $A_{2^9 \cdot 3^2}$, then $\mathrm{Aut}(\Gamma) = T$.*