European Journal of Combinatorics 63 (2017) 164–175

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On well quasi-order of graph classes under homomorphic image orderings

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ARTICLE INFO

Article history:
Received 3 November 2016
Accepted 17 March 2017
Available online 7 April 2017

ABSTRACT

In this paper we consider the question of well quasi-order for classes defined by a single obstruction within the classes of all graphs, digraphs and tournaments, under the homomorphic image ordering (in both its standard and strong forms). The homomorphic image ordering was introduced by the authors in a previous paper and corresponds to the existence of a surjective homomorphism between two structures. We obtain complete characterisations in all cases except for graphs under the strong ordering, where some open questions remain.

1. Introduction

Combinatorial structures have been considered under various different orderings; for example, substructure order (for which we may make a further distinction between weak and induced) and homomorphism order. For specific types of combinatorial object, there are other well-known orderings, for example the minor order on the class of graphs. All of these have received considerable attention in the combinatorial literature.

The starting premise in this work is the observation that the notion of homomorphism provides a useful unifying viewpoint from which to consider many of these orderings. Two structures A and B are related under the homomorphism (quasi-)order if there exists *any* homomorphism between them, while A and B are related under the substructure order if there exists an *injective* homomorphism between them (a "standard" homomorphism in the case of weak substructure, and a strong homomorphism in the case of induced substructure). The study of substructure orderings is perva-

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http://dx.doi.org/10.1016/j.ejc.2017.03.008

sive throughout combinatorics; for an introduction into the homomorphism ordering for graphs the reader may refer to [6, Chapter 3].

By way of analogy with the substructure order, it is natural to consider the partial order corresponding to the existence of a *surjective* homomorphism between two structures. As with the substructure order, we may distinguish between weak and induced forms. In a previous paper [8], we introduced this order, which we called the *homomorphic image order*; consideration of different strengths led to the standard, strong and M-strong forms of the order. Perhaps surprisingly, this order had previously received very little attention in the literature. One notable exception is [13], where the homomorphic image ordering is considered for the class of countable linear orders. As well as the naturalness of the definition, another motivation for studying the homomorphic image orders is that the graph minor order may be viewed as a composition of substructure order with a special kind of homomorphic image order. With the recent increase in prominence of minor-like orders (see, for example, [2,10]), one might hope that further understanding of homomorphic image orders for graphs could enable new insights into the minor orders.

Some fundamental graph-theoretical properties are preserved by the taking of homomorphic images, for instance being connected, and having diameter at most d. In particular, the class of all connected graphs can be defined by avoiding (in the sense of homomorphic images) the empty graph of size 2. It is perhaps worth noting that the above properties are not preserved by taking subgraphs (standard or induced), while the properties that are known to be preserved by the latter, such as planarity, are not preserved by homomorphic images.

This duality carries through into the area of well quasi-order and antichains; it transpires that the properties of the homomorphic image order are quite different in flavour from those of the more familiar substructure order. Within the class of (reflexive) graphs, many of the "classic" antichains under the substructure order, for example cycles and double-ended forks, are not antichains under the homomorphic image order (both of these in fact become chains). Conversely, antichains under the homomorphic image order may not be antichains in the substructure order; for example, the family of complete graphs with alternate perimeter edges deleted, forms an antichain under the former but not under the latter.

Well quasi-order for classes of graphs and related combinatorial structures is a natural and much-studied topic. Whenever we have classes of such structures which we wish to compare, for example in terms of inclusion or homomorphic images, we are led to consider downward-closed sets under the chosen orderings. The concept of well quasi-order then allows us to distinguish between what we may call (following Cherlin in [3]) "tame" and "wild" such classes.

A *quasi-order* is a binary relation which is reflexive ($x \leq x$ for all x) and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). A quasi-order which is also anti-symmetric ($x \leq y$ and $y \leq x$ implies $x = y$) is called a *partial order*; all orders considered in this paper are partial orders.

A *well quasi-order (wqo)* is a quasi-order which is *well-founded*, i.e. every strictly decreasing sequence is finite, and *has no infinite antichain*, i.e. every set of pairwise incomparable elements is finite. Since we will be considering only finite structures, and our orderings respect size, wqo is equivalent to the non-existence of infinite antichains throughout.

Given a quasi-order $(X, \leq)$, a subset I of X is called an *ideal* or *downward closed set* if $y \leq x \in I$ implies $y \in I$. Ideals are precisely *avoidance sets*, i.e. sets of the form $\mathrm{Av}(B) = \{x \in X : (\forall b \in B)(b \not\leq x)\}$. Here B is an arbitrary subset of X, finite or infinite. The situation in which every ideal is defined by a *finite* avoidance set is precisely the case when X is wqo.

Questions about well quasi-order of graphs and related structures have been extensively investigated. While the class of all graphs is not wqo under the subgraph order nor the induced subgraph order, a celebrated result of Robertson and Seymour [15] establishes that it is wqo under the minor order. When a class itself is not wqo, one can investigate the wqo ideals within that class and attempt to describe them. For example, a result due to Ding [5] establishes that an ideal of graphs with respect to the subgraph ordering is wqo precisely if it contains only finitely many cycles and double-ended forks.

We may ask the following general question about a class $(\mathcal{C}, \leq)$ of finite structures equipped with a natural ordering: *given a finite set* $\{X_1, \ldots, X_k\} \subseteq \mathcal{C}$ *of forbidden structures, is the ideal* $\mathrm{Av}(X_1, \ldots, X_k)$ *wqo?*