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On the number of planar Eulerian orientations

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ABSTRACT

The number of planar Eulerian maps with n edges is well-known to have a simple expression. But what is the number of planar Eulerian *orientations* with n edges? This problem appears to be a difficult one. To approach it, we define and count families of subsets and supersets of planar Eulerian orientations, indexed by an integer k, that converge to the set of all planar Eulerian orientations as k increases. The generating functions of our subsets can be characterized by systems of polynomial equations, and are thus algebraic. The generating functions of our supersets are characterized by polynomial systems involving divided differences, as often occurs in map enumeration. We prove that these series are algebraic as well. We obtain in this way lower and upper bounds on the growth rate of planar Eulerian orientations, which appears to be around 12.5.

1. Introduction

The enumeration of planar maps (graphs embedded on the sphere) has received a lot of attention since the sixties. Many remarkable counting results have been discovered, which were often illuminated later by beautiful bijective constructions. For instance, it has been known[1] since 1963 that the number of rooted planar *Eulerian* maps (i.e., planar maps in which every vertex has even

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[1] In disguise! The 1963 result involves *bicubic* maps, which are in one-to-one correspondence with Eulerian maps. See e.g. [15, Cor. 2.4] for the dual bijection between face-bicoloured triangulations and bipartite maps.

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Fig. 1. A rooted Eulerian map and a rooted Eulerian orientation.

degree) with n edges is [54]:

$$m_n = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}. \tag{1}$$

A bijective explanation involving plane trees can be found in [15]. The associated generating function $M(t) = \sum_{n \geq 0} m_n t^n$ is known to be *algebraic*, that is, to satisfy a polynomial equation. More precisely:

$$t^2 + 11t - 1 - (8t^2 + 12t - 1)M(t) + 16t^2 M(t)^2 = 0.$$

Beyond their enumerative implications, bijections involving maps have been applied to encode, sample and draw maps efficiently [11,21,31,50]. More recently, they have played a key role in the study of large random planar maps, culminating with the existence of a universal scaling limit known as the *Brownian map* [42].

Planar maps equipped with an additional structure (e.g. a spanning tree [43], a proper colouring [56,57], an Ising or Potts configuration [4,12,13,16,18,22,25,39]...) are also much studied, both in combinatorics and in theoretical physics, where maps are considered as a model for two-dimensional quantum gravity [23]. However, for many of these structures, we are still in the early days of the study, as even their enumeration remains elusive, not to mention bijections and asymptotic properties.

Recent progresses in this direction include the enumeration of planar maps weighted by their Tutte polynomial, or equivalently, maps equipped with a Potts configuration. The associated generating function $P(t)$ is known to be *differentially algebraic*. That is, there exists a polynomial equation relating $P(t)$ and its derivatives [6,7]. The Tutte polynomial has many interesting specializations (in particular, it counts all structures cited above, like spanning trees and colourings) and several special cases had been solved earlier. One key tool in the solution is that the Tutte polynomial of a map can be computed inductively, by deleting and contracting edges.

Another solved example, which does not seem to belong to the Tutte/Potts realm, consists of maps (in fact, triangulations) equipped with certain orientations called *Schnyder orientations*. The results obtained there have analogies with those obtained for another class of orientations, called *bipolar*, (which *do* belong to the Tutte realm). Indeed, for both classes of oriented maps:

- oriented maps are counted by simple numbers, which are also known to count other combinatorial objects (various lattice paths and permutations, among others);
- there exist nice bijections explaining these equi-enumeration results [9,10,28,33];
- for a fixed map M, the set of Schnyder/bipolar orientations of M has a lattice structure [51,27,45]. The above bijections, once specialized to maps equipped with their (unique) minimal orientation, coincide with attractive bijections designed earlier for (unoriented) maps [5,10];
- specializing the bijections further to maps that have only one Schnyder/bipolar orientation also yields interesting combinatorial results [5,10].

These observations led us to wonder about another natural class of orientations, namely those in which every vertex has equal in- and out-degree, known as *Eulerian* orientations (Fig. 1). Clearly, a map needs to be Eulerian to admit an Eulerian orientation. The condition is in fact sufficient (such maps even admit an Eulerian circuit [37]). One analogy with the above two classes is that the set of Eulerian orientations of a given planar map can be equipped with a lattice structure [51,27]. Moreover, Eulerian maps (equivalently, Eulerian maps equipped with their minimal Eulerian orientation) have rich combinatorial properties: not only are they counted by simple numbers (see (1)), but they are