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Packing and covering immersion-expansions of planar sub-cubic graphs[☆]

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ABSTRACT

A graph H is an immersion of a graph G if H can be obtained by some subgraph G after lifting incident edges. We prove that there is a polynomial function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that if H is a connected planar sub-cubic graph on $h > 0$ edges, G is a graph, and k is a non-negative integer, then either G contains k vertex/edge-disjoint subgraphs, each containing H as an immersion, or G contains a set F of $f(k, h)$ vertices/edges such that $G \setminus F$ does not contain H as an immersion.

1. Introduction

All graphs in this paper are finite, undirected, loopless, and may have multi-edges. Whenever we call a graph *simple* we also assume that it does not have multi-edges. Let $\mathcal{C}$ be a class of graphs. A *$\mathcal{C}$-vertex/edge cover* of G is a set S of vertices/edges such that each subgraph of G that is isomorphic to a graph in $\mathcal{C}$ contains some element of S. A *$\mathcal{C}$-vertex/edge packing* of G is a collection of vertex/edge-disjoint subgraphs of G, each isomorphic to some graph in $\mathcal{C}$.

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We say that a graph class $\mathcal{C}$ has the *vertex/edge Erdős–Pósa property* (shortly *v/e-E &P property*) for some graph class $\mathcal{G}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$, called a *gap function*, such that, for every graph G in $\mathcal{G}$ and every non-negative integer k, either G has a vertex/edge $\mathcal{C}$-packing of size k or G has a vertex/edge $\mathcal{C}$-cover of size $f(k)$. In the case where $\mathcal{G}$ is the class of all graphs we simply say that $\mathcal{C}$ has the v/e-E &P property. An interesting topic in Graph Theory, related to the notion of duality between graph parameters, is to detect instantiations of $\mathcal{C}$ and $\mathcal{G}$ such that $\mathcal{C}$ has the v/e-E &P property for $\mathcal{G}$ and, optimize the corresponding gap. Certainly, the first result of this type was the celebrated result of Erdős and Pósa in [11] who proved that the class of all cycles has the v-E &P property with gap function $O(k \cdot \log k)$. This result has triggered a lot of research on its possible extensions. One of the most general ones was given in [24] where it was proven that the class of graphs that are contractible to some graph H has the v-E &P property iff H is planar (see also [4,5,8] for improvements on the gap function).

Other instantiations of $\mathcal{C}$ for which the v-E &P property has been proved concern odd cycles [18,21], long cycles [2], and graphs containing cliques as minors [9] (see also [14,16,23] for results on more general combinatorial structures).

As noticed in [8], cycles have the e-E &P property as well. Interestingly, only few more results exist for the cases where the e-E &P property is satisfied. It is known for instance that graphs contractible to θ_r (i.e. the graph consisting of two vertices and an edge of multiplicity r between them) have the e-E &P property [3]. Moreover it was proven that odd cycles have the e-E &P property for planar graphs [19] and for 4-edge-connected graphs [18].

Given two graphs G and H, we say that H is an *immersion* of G if H can be obtained from some subgraph of G by lifting incident edges (see Section 2 for the definition of the lift operation). Given a graph H, we denote by $\mathcal{I}(H)$ the set of all graphs that contain H as an immersion. Using this terminology, the edge variant of the original result of Erdős and Pósa in [11] implies that the class $\mathcal{I}(\theta_2)$ has the v-E &P property (and, according to [8], the e-E &P property as well). A natural question is whether this can be extended for $\mathcal{I}(H)$, for other graphs H, different than θ_2. This is the question that we consider in this paper. A distinct line of research is to identify the graph classes $\mathcal{G}$ such that for every graph H, $\mathcal{I}(H)$ has the e-E &P property for $\mathcal{G}$. In this direction, it was recently proved in [20] that for every graph H, $\mathcal{I}(H)$ has the e-E &P property for 4-edge-connected graphs.

In this paper we show that if H is non-trivial (i.e., has at least one edge), connected, planar, and sub-cubic, i.e., each vertex is incident with at most 3 edges, then $\mathcal{I}(H)$ has the v/e-E &P property (with polynomial gap in both cases). More concretely, our main result is the following.

Theorem 1. *Let $k \in \mathbb{N}$. If H is a connected planar sub-cubic graph and G is a graph without any $\mathcal{I}(H)$-vertex/edge packing of size greater than k then G has a $\mathcal{I}(H)$-vertex/edge cover of size bounded by a polynomial function of h and k, where $h = |E(H)|$.*

The main tools of our proof are the graph invariants of tree-cut width and tree-partition width, defined in [28] and [10] respectively (see Section 2 for the formal definitions). Our proof uses the fact that every graph of polynomially (on k) big tree-cut width contains a wall of height k as an immersion (as proved in [28]). This permits us to consider only graphs of bounded tree-cut width and, by applying suitable reductions, we finally reduce the problem to graphs of bounded tree-partition width (Theorem 2). The result follows as we next prove that for every H, the class $\mathcal{I}(H)$ has the e-E &P property for graphs of bounded tree-partition width (Theorem 3).

One might conjecture that the result in Theorem 1 is tight in the sense that both being planar and sub-cubic are necessary for H in order for $\mathcal{I}(H)$ to have the e-E &P property. In this direction, in Section 7, we give counterexamples for the cases where H is planar but not sub-cubic and is sub-cubic but not planar.

2. Definitions and preliminary results

We use $\mathbb{N}^+$ for the set of all positive integers and we set $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$. Given a function $f : A \to B$ and a set $C \subseteq A$, we denote by $f|_C = \{(x, f(x)) \mid x \in C\}$.