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Cycles through all finite vertex sets in infinite graphs[☆]

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ABSTRACT

A closed curve in the Freudenthal compactification $|G|$ of an infinite locally finite graph G is called a *Hamiltonian curve* if it meets every vertex of G exactly once (and hence it meets every end at least once). We prove that $|G|$ has a Hamiltonian curve if and only if every finite vertex set of G is contained in a cycle of G. We apply this to extend a number of results and conjectures on finite graphs to Hamiltonian curves in infinite locally finite graphs. For example, Barnette's conjecture (that every finite planar cubic 3-connected bipartite graph is Hamiltonian) is equivalent to the statement that every one-ended planar cubic 3-connected bipartite graph has a Hamiltonian curve. It is also equivalent to the statement that every planar cubic 3-connected bipartite graph with a nowhere-zero 3-flow (with no restriction on the number of ends) has a Hamiltonian curve. However, there are 7-ended planar cubic 3-connected bipartite graphs that do not have a Hamiltonian curve.

1. Introduction

Diestel [6] launched the ambitious project of extending results on finite Hamiltonian cycles to Hamiltonian circles, that is simple closed curves traversing each vertex and each end precisely once

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in the Freudenthal compactification $|G|$ of a locally finite graph G. Diestel specifically suggested to find Hamiltonian circles in the square of a graph in order to obtain a unification of Fleischner's theorem [11] and its counterpart for one-ended graphs by Thomassen [28]. This was done by Georgakopoulos [13].

Georgakopoulos [14] (see also [6]) then conjectured that every 4-connected line graph has a Hamiltonian circle. Bruhn (see [6]) conjectured that Tutte's theorem on Hamiltonian cycles in 4-connected planar graphs can be extended to Hamiltonian circles. These conjectures are open but significant progress has been made by Lehner [24] on the former and by Bruhn and Yu [3] on the latter. Several other results in this area can be found in [5,17,21–23].

In the present paper we consider another natural way of extending results on finite Hamiltonian cycles to infinite locally finite graphs, namely the following property: for every finite vertex set S, the graph has a cycle containing S. We prove that this property is related to the Hamiltonian circle problem in that the property holds if and only if the Freudenthal compactification $|G|$ has a Hamiltonian curve, that is, a simple closed curve traversing each vertex precisely once (and hence each end at least once). The most difficult part here is the implication from infinite to finite graphs. For a finite vertex set S we let G_S be the graph obtained from $G[S]$, the subgraph induced by S, by adding an edge between each pair of vertices that are joined by a path internally disjoint from S. If G has a cycle whose vertex set contains S, then clearly G_S has a Hamiltonian cycle. The converse is not true for a fixed S. However, our result implies that, if G_S has a Hamiltonian cycle for each finite vertex set S, then also G has a cycle containing S, for each finite S. As the proof goes via Hamiltonian curves, one may see this as a way of using infinite substructures to find finite substructures.

As pointed out by Georgakopoulos [13], we can often go in the other direction and obtain Hamiltonian curves by appropriate ad hoc compactness arguments. Our characterization tells more precisely when that can be done, and we illustrate this by extending various results and problems on finite graphs. A particularly interesting example is Barnette's conjecture that every finite planar bipartite cubic 3-connected graph has a Hamiltonian cycle. An infinite planar bipartite cubic 3-connected graph need not have a Hamiltonian curve. Indeed, there are 7-ended counterexamples. Instead we show that Barnette's conjecture for finite graphs is equivalent with the following: Every infinite planar cubic 3-connected graph having a nowhere-zero 3-flow has a Hamiltonian curve. Using this we also conclude that Barnette's conjecture for finite graphs is also equivalent with the following: Every infinite planar cubic 3-connected bipartite one-ended graph has a Hamiltonian curve.

2. Notation and terminology

All graphs are *locally finite* in this paper, that is, all vertices have finite degree. We follow Diestel [8] in our basic terminology for infinite graphs G. Specifically, a *ray* is a one-way infinite path, and two such rays in a graph G are *equivalent* if for every finite vertex set S both rays have a tail in the same component of $G-S$. An *end* α of G is an equivalence class of rays, and this can be viewed as a particular "point at infinity". For a finite vertex set S we now denote the unique component of $G-S$ containing a tail of every ray in α by $C(S,\alpha)$. We also let $\Omega(G)$ be the set of ends of G, and $\Omega(S,\alpha)$ be the set of all ends β with $C(S,\beta)=C(S,\alpha)$.

To define a topology on G we associate each edge uv with a homeomorphic image of $[0,1]$ where 0,1 map to u, v (and where different edges may only intersect at common endpoints). Basic open neighborhoods of points that are vertices or contained in edges are defined in the usual way, that is we start with the basic open neighborhoods in the topology of the 1-complex. For an end α we proceed as follows: For every finite vertex set S we let the basic neighborhood $\widehat{C}(S,\alpha)$ consist of $C(S,\alpha)$, $\Omega(S,\alpha)$ and the edges between $C(S,\alpha)$ and S (except their endvertices in S). It is not hard to verify that a connected locally finite graph G together with its ends $\Omega(G)$ is a compact Hausdorff space in this topology called the *Freudenthal compactification* $|G|$ of G (see [8] Proposition 8.6.1). The closure of $C(S,\alpha)$ in this topology is $C(S,\alpha)\cup\Omega(S,\alpha)$.

Let $\boldsymbol{x}=(x_i)_{i=1}^{\infty}$ be a vertex sequence of G. We say that a set A contains *almost all* terms of $\boldsymbol{x}$ if there exists j such that $x_i\in A$ for all $i\geq j$. So the sequence $\boldsymbol{x}$ converges to an end $\alpha\in\Omega(G)$ if every $C(S,\alpha)$ for S finite contains almost all terms of $\boldsymbol{x}$. Thus the closure of a vertex set X is $X\cup\Omega(X)$, where $\Omega(X)$ denotes the set of ends of G such that there is a sequence in X converging to the end. Conversely, $\alpha\in\Omega(G)-\Omega(X)$ if and only if there is a finite vertex set S such that $C(S,\alpha)$ is disjoint from X.