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Counting results for sparse pseudorandom hypergraphs I

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ABSTRACT

We establish a so-called counting lemma that allows embeddings of certain linear uniform hypergraphs into sparse pseudorandom hypergraphs, generalizing a result for graphs (Kohayakawa et al., 2004). Applications of our result are presented in the companion paper (Counting results for sparse pseudorandom hypergraphs II).

1. Introduction

Many problems in extremal combinatorics concern embeddings of graphs and hypergraphs of fixed isomorphism type into a large host graph/hypergraph. The systematic study of pseudorandom graphs was initiated by Thomason [15,16] and since then many embedding results have been developed for host pseudorandom graphs. For example, a well-known consequence of the Chung–Graham–Wilson theorem [4] asserts that dense pseudorandom graphs G contain the "right" number of copies of any fixed graph, where "right" means approximately the same number of copies as expected in a random graph with the same density as G. In view of this result, the question arises to which extent it can be generalized to sparse pseudorandom graphs, and results in this direction can be found in [2,3,5,12]. We continue this line of research for embedding properties of sparse pseudorandom hypergraphs. Counting lemmas for pseudorandom hypergraphs were also investigated by Conlon, Fox and Zhao [6].

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Let $G = (V, E)$ be a k-uniform hypergraph. For every $1 \leqslant i \leqslant k-1$ and every i-element set $\{x_1, \dots, x_i\} \in \binom{V}{i}$, let

$$N_G(x_1, \dots, x_i) = \left\{ \{x_{i+1}, \dots, x_k\} \in \binom{V}{k-i} : \{x_1, \dots, x_k\} \in E \right\},$$

i.e., $N_G(x_1, \dots, x_i)$ is the set of elements of $\binom{V}{k-i}$ that form an edge of G together with $\{x_1, \dots, x_i\}$. In what follows, our hypergraphs will usually be k-uniform and will have n vertices. The parameters n and k will often be omitted if there is no danger of confusion.

Property 1.1 (*Boundedness Property*). *Let $k \geqslant 2$. We say that an n-vertex k-uniform hypergraph $G = (V, E)$ satisfies* BDD(d, C, p) *if, for all $1 \leqslant r \leqslant d$ and all families of distinct sets $S_1, \dots, S_r \in \binom{V}{k-1}$, we have*

$$|N_G(S_1) \cap \dots \cap N_G(S_r)| \leqslant Cnp^r. \tag{1}$$

Property 1.2 (*Tuple Property*). *Let $k \geqslant 2$. We say that an n-vertex k-uniform hypergraph $G = (V, E)$ satisfies* TUPLE(d, δ, p) *if, for all $1 \leqslant r \leqslant d$, the following holds.*

$$\big| |N_G(S_1) \cap \dots \cap N_G(S_r)| - np^r \big| < \delta np^r \tag{2}$$

for all but at most $\delta \binom{\binom{n}{k-1}}{r}$ families $\{S_1, \dots, S_r\}$ of r distinct sets of $\binom{V}{k-1}$.

The notion of pseudorandomness considered in this paper is given in Definition 1.3.

Definition 1.3. A k-uniform hypergraph $G = (V, E)$ is (d_1, C, d_2, δ, p)-*pseudorandom* if $|E| = p\binom{n}{k}$ and G satisfies BDD(d_1, C, p) and TUPLE(d_2, δ, p).

Note that property TUPLE implies edge-density close to p, but we put the condition $|E| = p\binom{n}{k}$ in the definition of pseudorandomness for convenience. We remark that similar notions of pseudorandomness in hypergraphs were considered in [8,9].

A hypergraph H is called *linear* if every pair of edges shares at most one vertex. Our main result, Theorem 1.4, estimates the number of copies of some linear k-uniform hypergraphs in sparse pseudorandom hypergraphs. An *embedding* of a hypergraph H into a hypergraph G is an injective mapping $\phi\colon V(H) \to V(G)$ such that $\{\phi(v_1), \dots, \phi(v_k)\} \in E(G)$ whenever $\{v_1, \dots, v_k\} \in E(H)$. An edge e of a linear k-uniform hypergraph $E(H)$ is called *connector* if there exist $v \in V(H) \setminus e$ and k edges $e_1, \dots, e_k$ containing v such that $|e \cap e_i| = 1$ for $1 \leqslant i \leqslant k$. Note that, for $k = 2$, a connector is an edge that is contained in a triangle. Moreover, since H is linear, $e \cap e_i \neq e \cap e_j$ for all $1 \leqslant i < j \leqslant k$. Given a k-uniform hypergraph H, let

$$d_H = \max\{\delta(J) : J \subset H\} \quad \text{and} \quad D_H = \min\{kd_H, \Delta(H)\},$$

where $\delta(J)$ and $\Delta(J)$ stand, respectively, for the minimum and the maximum degree of a vertex in $V(J)$. Note that $d_H \leqslant D_H$.

Kohayakawa, Rödl and Sissokho [12] proved the following counting lemma: *given a fixed triangle-free graph H and $p = p(n) \gg n^{-1/D_H}$ with $p = o(1)$, for all $\varepsilon > 0$ and $C > 1$, there exists $\delta > 0$ such that, if G is an n-vertex $(D_H, C, 2, \delta, p)$-pseudorandom graph and n is sufficiently large, then*

$$\big| |\mathcal{E}(H, G)| - n^{v(H)}p^{e(H)} \big| < \varepsilon n^{v(H)}p^{e(H)},$$

where $\mathcal{E}(H, G)$ stands for the set of all embeddings from H into G. The triangle-freeness condition on H is necessary and the reader is referred to [12] for a detailed discussion of this fact. Our main theorem generalizes this result for k-uniform hypergraphs.