European Journal of Combinatorics 62 (2017) 58–69

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Algebraic flow theory of infinite graphs

B. Miraftab [a], M.J. Moghadamzadeh [b]

[a] Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany
[b] Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

ARTICLE INFO

Article history:
Received 26 January 2016
Accepted 21 November 2016

ABSTRACT

A problem by Diestel is to extend algebraic flow theory of finite graphs to infinite graphs with ends. In order to pursue this problem, we define an A-flow and non-elusive H-flow for arbitrary graphs and for abelian Hausdorff topological groups H and compact subsets $A \subseteq H$. We use these new definitions to extend several well-known theorems of flows in finite graphs to infinite graphs.

1. Introduction

The concept of flow is a main topic in graph theory and has various applications, as e.g. in electric networks. Algebraic flow theory for finite graphs is well studied, see [10–12,14,17]. But when it comes to infinite graphs, much less is known. There are some results for electrical networks, see [1,7–9], but not for group-valued flows. In fact Diestel's problem [8, Problem 4.27] to extend flow theory to infinite graphs is still widely open. Here we are doing a first step toward its solution.

In Section 2, we give our main definition for flows in infinite graphs. Roughly speaking, a flow is a map from the edge set of a graph to an abelian Hausdorff topological group such that the sum over all edges in each finite cut is trivial. With this in mind, we shall extend the following theorems of finite graphs:

- A finite graph has a non-elusive $\mathbb{Z}_2$-flow if and only if its degrees are even.
- A finite cubic graph has a non-elusive $\mathbb{Z}_4$-flow if and only if it is 3-edge-colorable.
- Every finite graph containing a Hamilton cycle has a non-elusive $\mathbb{Z}_4$-flow.

Our main tool to prove these results is Theorem 5, which offers some kind of compactness method to extend results for finite graphs to infinite graphs of arbitrary degree, i.e. that need not be locally finite. However it is worth remarking that not all theorems about flows in finite graphs have a

E-mail addresses: babak.miraftab@uni-hamburg.de (B. Miraftab), javad_mz123@yahoo.com (M.J. Moghadamzadeh).

http://dx.doi.org/10.1016/j.ejc.2016.11.009
0195-6698/

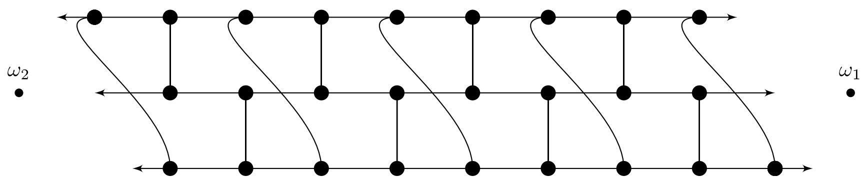

Fig. 1.1. The cubic bipartite graph without any non-elusive $\mathbb{Z}_3$-flow.

straightforward analogue in the infinite case: a finite cubic graph G has a non-elusive $\mathbb{Z}_3$-flow if and only if G is bipartite, see [7, Proposition 6.4.2]. This is wrong for infinite graphs. Fig. 1.1 shows a cubic bipartite graph without any non-elusive $\mathbb{Z}_3$-flow. Even further restrictions on the ends of that graph, e.g. requiring them to have edge- or vertex-degree 3, fails in our example. (For more about the ends of a graph and the topological setting, we refer readers to [8] and the references therein.)

In Section 4, we define the concept of tension for infinite graphs. Heuristically, a tension is a map from the edge set of a graph to an abelian Hausdorff topological group such that the sum over all edges in each finite cycle is trivial.

2. Preliminaries

We refer readers to [7], for the standard terminology and notations in this paper. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its tails. Two rays in a graph $G = (V, E)$ are equivalent if no finite set of vertices separates them. This is an equivalence relation whose classes are the *ends* of G. Now, consider a locally finite graph G as one-dimensional CW complex and compactify G by using the Freudenthal compactification method. We denote this new topological space by $|G|$, for more on $|G|$, see [6] and [8]. Let D be a subset of edges of G. We denote the closure of the point set $\cup_{d\in D} d$ in $|G|$ by $\overline{D}$. A *circle* in $|G|$ is a homeomorphic image of the unit circle S^1. Analogously an *arc* in $|G|$ is a homeomorphic image of the closed interval $[0, 1]$. We denote the *cut space*, *finite cut space*, *topological cycle space* and *finite cycle space* of a graph G by $\mathcal{B}(G)$, $\mathcal{B}_{\text{fin}}(G)$, $\mathcal{C}(G)$ and $\mathcal{C}_{\text{fin}}(G)$, respectively. For more details about the equivalent definitions of topological cycle space and its properties, see [7,8]. Note that $\mathcal{B}(G)$ is a vector space over $\mathbb{Z}_2$. We now define the degree of an end of the graph G. The *edge-degree* of an end ω is the maximum number of edge-disjoint rays in ω. In addition, let D be a subset of the set of edges of G. Then we say that an end ω is *D-even* if there exists a finite vertex set S so that for all finite vertex sets $S' \supseteq S$ it holds that the maximal number of edge-disjoint arcs from S' to ω contained in $\overline{D}$ is even. If D is all the edges of G, we remove D from the notation and we only say that ω has an even edge-degree. For more about the degree of ends, see [3,4]. The following theorem describes the elements of the cycle space for locally finite graphs. For the proof, see [7, Theorem 8.5.10] and [2, Theorem 5].

Theorem 1. *Let $G = (V, E)$ be a locally finite connected graph. Then an edge set $D \subseteq E$ lies in $\mathcal{C}(G)$ if and only if one of the following equivalent statements holds*

(i) *D meets every finite cut in an even number of edges.*
(ii) *Every vertex and every end of G is D-even.*

Let us review some notions of the compactness method for locally finite graphs. Suppose that $v_0, v_1, \dots$ is an enumeration of V. We define $S_n = v_0, \dots, v_n$, for every $n \in \mathbb{N}$. Put G_n for the minor of G obtained by contracting each component of $G \setminus S_n$ to a vertex. Note that we delete any loop, but we keep multiple edges. The vertices of G_n outside S_n are called *dummy vertices* of G_n. Let $G = (V, E)$ be a graph. A *directed edge* is an ordered triple (e, x, y), where $e = xy \in E$. So we can present each edge according to its direction by $\overrightarrow{e} = (e, x, y)$ or $\overleftarrow{e} = (e, y, x)$. We use $\overrightarrow{E}$ for the set of all oriented edges of G. For two subsets X, Y (not necessarily disjoint) of V and a subset $\overrightarrow{C}$ of $\overrightarrow{E}$, we define

$$\overrightarrow{C}(X, Y) := \{(e, x, y) \in \overrightarrow{C} \mid x \in X, y \in Y, x \neq y\}.$$