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Counting triangulations of some classes of subdivided convex polygons

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ABSTRACT

We compute the number of triangulations of a convex k-gon each of whose sides is subdivided by $r-1$ points. We find explicit formulas and generating functions, and we determine the asymptotic behavior of these numbers as k and/or r tend to infinity. We connect these results with the question of finding the planar set of points in general position that has the minimum possible number of triangulations — a well-known open problem from computational geometry.

1. Introduction

Let k and r be two natural numbers, $k \geq 3$, $r \geq 1$. Let $\mathrm{SC}(k, r)$ denote a convex k-gon in the plane each of whose sides is subdivided by $r-1$ points. (Thus, the whole configuration consists of kr points.) In what follows, the exact measures are not essential: without loss of generality, we may consider a regular k-gon with sides subdivided by evenly spaced points. The k vertices of the original ("basic") k-gon will be called *corners*, and they will be denoted (say, clockwise) by $P_{0,0}, P_{1,0}, \dots, P_{k-1,0}$ (with arithmetic modulo k in the first index, so that $P_{k,0} = P_{0,0}$). The $r-1$ points that subdivide the segment $P_{i,0}P_{i+1,0}$ (oriented from $P_{i,0}$ to $P_{i+1,0}$) will be denoted by $P_{i,1}, P_{i,2}, \dots, P_{i,r-1}$ (we shall also occasionally

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0195-6698/

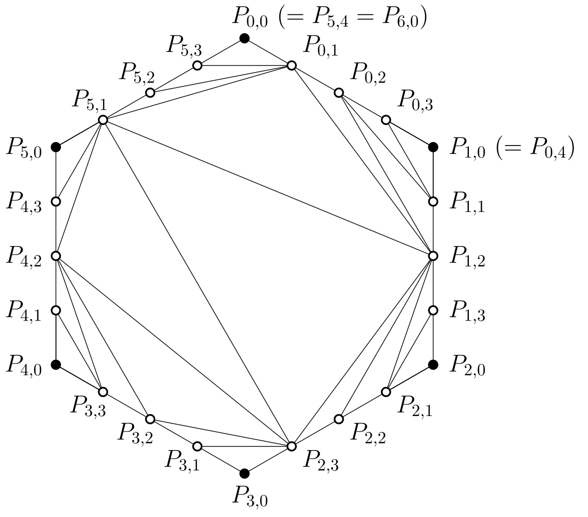

Fig. 1. The subdivided convex polygon SC(6, 4) and one of its triangulations.

write $P_{i,r}$ for $P_{i+1,0}$). The subdivided segments $P_{i,0}P_{i+1,0}$ – that is, the point sequences of the form $P_{i,0}, P_{i,1}, P_{i,2}, \ldots, P_{i,r-1}, P_{i+1,0}$ – will be referred to as *strings*. Thus, the boundary of $\mathsf{SC}(k, r)$ consists of k strings, and each corner belongs to two strings. The reader is referred to Fig. 1 for an illustration. For brevity, a convex polygon with subdivided edges (not all of them necessarily subdivided by the same number of points) will be referred to as a *subdivided convex polygon*. A subdivided convex polygon is *balanced* if (as described above) all its sides are subdivided by the same number of points.

A *triangulation* of a finite planar point set S is a dissection of its convex hull by non-crossing diagonals[1] into triangles. We emphasize that maximal triangulations are meant; in particular, no triangle can have another point of the set in the interior of one of its sides. The set of triangulations of a point set S will be denoted by $\mathsf{TR}(S)$.

Triangulations of (structures equivalent or related to) subdivided convex polygons have appeared in earlier work. Hurtado and Noy [11] considered triangulations of *almost convex polygons*, which turn out to be equivalent to subdivided convex polygons according to our terminology. They dealt with the non-balanced case—that is, k-gons whose sides are subdivided, but not necessarily into the same number of points. In particular, Hurtado and Noy derived an inclusion–exclusion formula for the number of triangulations of a subdivided convex k-gon whose sides are subdivided by $a_1, a_2, \ldots, a_k$ points, and they showed that this number is independent of the specific distribution of the subdivisions among the sides of the basic k-gon. On the other hand, Bacher and Mouton [6,7] considered triangulations of more general *nearly convex polygons* defined as infinitesimal perturbations of subdivided convex polygons. They derived a formula for the number of triangulations of such polygons in terms of certain polynomials that depend on the shape of chains.

The main purpose of our paper is to present enumeration formulas and precise asymptotic results for the number of triangulations of a subdivided convex polygon in the balanced case, that is, where each side of the polygon is subdivided into the same number of points. Our enumeration formulas are more compact than those of Hurtado and Noy or of Bacher and Mouton when specialized to the balanced case. We shall as well provide formulas for some non-balanced cases.

Let us denote the number of triangulations of $\mathsf{SC}(k, r)$ by $\mathsf{tr}(k, r)$. For $r = 1$ our configuration is just a convex k-gon, and, thus, $\mathsf{tr}(k, 1) = C_{k-2}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. It is easy to find $\mathsf{tr}(k, r)$ for small values of k and r by inspection. For example, we have $\mathsf{tr}(3, 2) = 4$,

[1] By a "diagonal" we mean a straight-line segment connecting two points of the set S.