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On the Zero Defect Conjecture

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A B S T R A C T

Brlek et al., conjectured in 2008 that any fixed point of a primitive morphism with finite palindromic defect is either periodic or its palindromic defect is zero. Bucci and Vaslet disproved this conjecture in 2012 by a counterexample over ternary alphabet. We prove that the conjecture is valid on binary alphabet. We also describe a class of morphisms over multiliteral alphabet for which the conjecture still holds. The proof is based on properties of extension graphs.

1. Introduction

Palindromes – words read the same from the left as from the right – are a favorite pun in various languages. For instance, the words ressasser, t'ahat', and šílíš are palindromic words in the first languages of the authors of this paper. The reason for a study of palindromes in formal languages is not only to deepen the theory, but it has also applications.

The theoretical reasons include the fact that a Sturmian word, i.e., an infinite aperiodic word with the least factor complexity, can be characterized using the number of palindromic factors of given length that occur in a word, see [10]. The application motives include the study of the spectra of discrete Schrödinger operators, see [12,13].

In [9], the authors provide an elementary observation that a finite word of length n cannot contain more than $n+1$ (distinct) palindromic factors, including the empty word as a palindromic factor. We

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illustrate this on the following 2 examples of words of length 9:

$$w^{(1)} = 010010100 \quad \text{and} \quad w^{(2)} = 011010011.$$

The word $w^{(1)}$ is a prefix of the famous Fibonacci word and w_2 is a prefix of (also famous) Thue–Morse word. There are 10 palindromic factors of $w^{(1)}$: 0, 1, 00, 010, 101, 1001, 01010, 010010, 0010100, and the empty word. The word $w^{(2)}$ contains only 9 palindromes: 0, 1, 11, 0110, 101, 010, 00, 1001, and the empty word.

The existence of the upper bound on the number of distinct palindromic factors leads to the definition of *palindromic defect* (or simply *defect*) of a finite word w, see [5], as the value

$$D(w) = n + 1 - \text{the number of palindromic factors of } w$$

with n being the length of w. Our examples satisfy $D(w^{(1)}) = 0$, i.e., the upper bound is attained, and $D(w^{(2)}) = 1$. The notion of palindromic defect naturally extends to infinite words. For an infinite word $\mathbf{u}$ we set

$$D(\mathbf{u}) = \sup\{D(w) : w \text{ is a factor of } \mathbf{u}\}.$$

In this paper, we deal with infinite words that are generated by a primitive morphism of a free monoid $\mathcal{A}^*$ with $\mathcal{A}$ being a finite alphabet. A morphism φ is completely determined by the images of all letters $a \in \mathcal{A}$: $a \mapsto \varphi(a) \in \mathcal{A}^*$. A morphism is *primitive* if there exists a power k such that any letter $b \in \mathcal{A}$ appears in the word $\varphi^k(a)$ for any letter $a \in \mathcal{A}$.

The two mentioned infinite words can be generated using a primitive morphism. Consider the morphism φ_F over $\{0, 1\}^*$ determined by $0 \mapsto 01$ and $1 \mapsto 0$. By repeated application of φ_F, starting from 0, we obtain

$$0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto 01001010 \ldots.$$

Since $\varphi_F^n(0)$ is a prefix of $\varphi_F^{n+1}(0)$ for all $n \in \mathbb{N}$, there exists an infinite word $\mathbf{u}_F$, called the Fibonacci word, such that $\varphi_F^n(0)$ is its prefix for all n. Consider a natural extension of φ_F to infinite words, we obtain that $\mathbf{u}_F$ is a fixed point of φ_F since

$$\mathbf{u}_F = \varphi_F(\mathbf{u}_F) = \varphi_F(u_0u_1u_2\ldots) = \varphi_F(u_0)\varphi_F(u_1)\varphi_F(u_2)\ldots$$

where $u_i \in \{0, 1\}$.

Similarly, let φ_{TM} be a morphism determined by $0 \mapsto 01$ and $1 \mapsto 10$. By repeated application of φ_{TM}, starting again from 0, we obtain

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110 \ldots.$$

The infinite word having $\varphi_{TM}^n(0)$ as a prefix for each n is the Thue–Morse word, sometimes also called Prouhet–Thue–Morse word.

The present article focuses on palindromic defect of infinite words which are fixed points of primitive morphisms. In order for the palindromic defect of such an infinite word to be finite, the word must contain an infinite number of palindromic factors. This property is satisfied by the two mentioned words $\mathbf{u}_F$ and $\mathbf{u}_{TM}$. However, for their palindromic defect, we have $D(\mathbf{u}_F) = 0$, whilst $D(\mathbf{u}_{TM}) = +\infty$.

There exist fixed points $\mathbf{u}$ of primitive morphisms with $0 < D(\mathbf{u}) < +\infty$, but on a two-letter alphabet, only ultimately periodic words are known. In [5], examples of such words are given by Brlek, Hamel, Nivat and Reutenauer as follows: for any $k \in \mathbb{Z}$, $k \geq 2$ denote by z the finite word

$$z = 01^k01^{k-1}001^{k-1}01^k0.$$

Then the infinite periodic word z^ω has palindromic defect k. Of course, the periodic word z^ω is fixed by the primitive morphism $0 \mapsto z$, $1 \mapsto z$. In [4], the authors stated the following conjecture:

Conjecture (*Zero Defect Conjecture*). *If $\mathbf{u}$ is a fixed point of a primitive morphism such that $D(\mathbf{u}) < +\infty$, then $\mathbf{u}$ is periodic or $D(\mathbf{u}) = 0$.*