European Journal of Combinatorics 62 (2017) 147–166

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Hardness of computing clique number and chromatic number for Cayley graphs

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ARTICLE INFO

Article history:
Received 10 April 2015
Accepted 15 December 2016
Available online 11 January 2017

ABSTRACT

Computing the clique number and chromatic number of a general graph are well-known NP-Hard problems. Codenotti et al., (1998) showed that computing clique number and chromatic number are both NP-Hard problems for the class of circulant graphs. We show that computing clique number is NP-Hard for the class of Cayley graphs for the groups G^n, where G is any fixed finite group (e.g., cubelike graphs). We also show that computing chromatic number cannot be done in polynomial time (under the assumption NP $\neq$ ZPP) for the same class of graphs. Our presentation uses free Cayley graphs. The proof combines free Cayley graphs with quotient graphs and Goppa codes.

In his celebrated 1972 paper [9], Karp established the NP-Completeness of 21 combinatorial problems. Amongst those problems are the CLIQUE problem and the CHROMATIC NUMBER problem. CLIQUE takes a graph X and an integer k and decides whether X contains a clique of size k as a subgraph. CHROMATIC NUMBER takes a graph X and an integer k and decides whether there is a proper colouring of X using at most k colours.

The *clique number* of a graph X is the size of the largest clique contained in X, and is denoted by $\omega(X)$. Since deciding whether a general graph X contains a clique of size k is NP-Complete, the problem of computing $\omega(X)$ is NP-Hard. The *chromatic number* of a graph X is the smallest integer

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http://dx.doi.org/10.1016/j.ejc.2016.12.005
0195-6698/

k such that X has a proper k-colouring, and is denoted by $\chi(X)$. Again, since deciding whether a general graph X can be coloured properly using at most k colours is NP-Complete, computing $\chi(X)$ is NP-Hard.

Some of the graph theoretic problems in Karp's list become easier when restricted to a subclass of graphs. For instance, deciding whether a graph X has a subset of vertices with size k that covers all of the edges of X is NP-Complete. However, if X is bipartite the Hungarian Algorithm finds a minimum vertex cover of X in polynomial time. There are also subclasses of graphs for which the clique number and chromatic number are computable in polynomial time. For example, planar graphs have polynomial time computable clique numbers, and graphs with treewidth at most k have polynomial time computable chromatic numbers [1].

In 1998, Codenotti, Gerace, and Vigna [5] proved that computing clique number, and chromatic number, are NP-Hard when restricted to the class of circulant graphs (a *circulant* is a Cayley graph for a group $\mathbb{Z}_m$). Since circulants are Cayley graphs, they are vertex transitive. One might hope, or expect, that the assumption of vertex transitivity would confer some advantage when approaching computational problems on graphs. Codenotti et al.'s results show that this is not the case, and also raise the question of whether there are classes of Cayley graphs on which these problems become easier.

Our main results in this paper are analogues of Codenotti et al.'s hardness results for a different class of Cayley graphs. Our results apply to the class of Cayley graphs for the groups G^n, where G is any fixed finite group. When $G = \mathbb{Z}_2$, this is the class of *cubelike graphs*. We show that computing clique number for these graphs is an NP-Hard problem (Theorem 10.2). We also show that computing chromatic number for these graphs cannot be done in polynomial time under the assumption that NP $\neq$ ZPP (Theorem 11.3). (The assumption NP $\neq$ ZPP arises from our application of a result of Håstad [7]. We refer the reader to the description of the class ZPP given there, as well as the claim that the assumption ZPP $\neq$ NP is similar to the assumption P $\neq$ NP.)

We prove that computing clique number is NP-Hard by reducing computing the clique number of a general graph X to computing the clique number of a Cayley graph on a group G^n. The key to this reduction is providing a construction of a Cayley graph Γ from X so that $|\Gamma|$ is polynomially bounded in $|X|$, and so that $\omega(X)$ is easily computed from $\omega(\Gamma)$. We begin with a construction used by Babai and Sós [2] to embed graphs in Cayley graphs. This construction leads naturally to free Cayley graphs. We construct a free Cayley graph $G(X)$ from X so that the cliques in X can be recovered from the cliques in $G(X)$. To complete our construction we will quotient $G(X)$ over a suitably chosen linear code. Specifically, we will give a Goppa code that satisfies the desired properties.

To prove that computing chromatic number cannot be done in polynomial time we use a similar strategy to Codenotti et al. In [5], the chromatic number result is proven by reducing computing clique number for general graphs to computing chromatic number for circulant graphs. This reduction does not work for the Cayley graphs we consider. However, we adapt this approach to show that if chromatic number can be computed in polynomial time for Cayley graphs for the groups G^n, then for any graph X, the clique number of X can be approximated to within a constant factor in polynomial time. This completes the proof by an inapproximability result of Håstad [7].

Constructing our reductions using free Cayley graphs situates them in a more general framework. Free Cayley graphs are relatively new and unstudied objects (they appear in a recent paper by Beaudou, Naserasr and Tardif [3]). Our results rely on the clique structure of free Cayley graphs, and provide a new application of these objects.

This paper has roughly three parts. The first part gives the background necessary for our results and their proofs. Sections 1 and 2 contain some basic graph theory and coding theory. Sections 3–5 introduce free Cayley graphs. The second part contains the intermediate results needed for our reductions, and the proofs of our main results. Section 6 develops the relationship between cliques in a graph X and cliques in the free Cayley graph $G(X)$. Sections 7 and 8 focus on the groups $\mathbb{Z}_p$ for p prime. The main results of this paper are given in Sections 10 and 11. The last part consists of some additional observations. We consider how our construction can be applied to embeddings in Section 12, and we take a closer look at free Cayley graphs in Section 13.