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Enumeration of 4-regular one-face maps

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ABSTRACT

We obtain explicit formulas for enumerating rooted and unrooted 4-regular one-face maps on genus g surfaces. For rooted maps the result is combinatorially derived from Chapuy's vertex cutting bijection and has a simple sum-free form similar to analogous formulas for general and cubic one-face maps. To enumerate unrooted maps we apply the approach of Liskovets, Mednykh and Nedela of reducing the problem to counting rooted maps on orbifolds. We show that for 4-regular one-face maps the set of orbifolds to be considered has a simple description, which allows us to obtain the final formula in an explicit form of a sum over 3 integer indices.

1. Introduction

By a *one-face* (or *unicellular*) *map* M on a compact connected orientable surface which is fully characterized by its genus g, we will denote an embedding of a connected graph G, loops and multiple edges allowed, into a compact and oriented surface X, such that G is as a subset of X and its complement $X \setminus G$ is homeomorphic to a topological polygon. This complement is the only *face* of the map M which has k vertices (points on the surface X) and n edges (nonintersecting curves on the surface that have no common points other than the vertices of the graph).

4-regular maps play an important role in different fields of mathematics. They have been used for the knot problem in low-dimensional topology [10], for rectilinear embedding in VLSI [19], for the Gaussian Crossing Problem [19], for the folding of RNA interaction structures [3], and for enumerating some other kinds of maps. In its turn, one-face maps are extensively used in a number of instances in pure mathematics including finite type invariants of knots and links [4,9], the representation theory of Lie algebras [5], the geometry of moduli spaces of flat connections on surfaces [2], mapping class groups [1] and in applied mathematics including codifying the pairings among nucleotides in RNA

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molecules [13] and data structures analysis [7]. The present paper is devoted to the problem of enumerating 4-regular one-face maps or, equivalently, objects dual to these maps, namely 1-vertex quadrangulations of a surface of genus g.

The first work on the subject of enumerating rooted maps on genus g surfaces was the work of Walsh and Lehman [17]. Using Tutte's approach for enumerating planar maps [14,15], the authors derived a recurrence relation for the numbers of rooted maps and calculated the first terms of the corresponding sequences. In addition, that paper contained an explicit expression for the number $\varepsilon_g(n)$ of one-face maps with n edges on a surface of genus g, as well as a formula for the number of unicellular maps of genus g with predefined vertex degrees. However, the combinatorial sense of these formulas remained unclear. The first success in deriving similar results combinatorially was the work [6]. Chapuy used an original approach based on the reduction of a one-face map to a one-face map of a lower genus. He obtained a new recurrence relation for the numbers $\varepsilon_g(n)$ and gave an elegant combinatorial interpretation of it. In the same paper it was shown how to use this technique to enumerate some special kinds of maps, for example, cubic one-face maps. In the first part of the present work we apply this approach to enumerate rooted 4-regular one-face maps of genus g.

A general technique for counting unrooted planar maps up to orientation-preserving homeomorphisms was developed by Liskovets [11] in the early eighties. His approach reduces the enumerating problem for unrooted maps on the sphere to counting so-called quotient maps, i.e. maps on quotients of the sphere under a finite group of automorphisms. His ideas were further developed by Mednykh and Nedela in a series of papers devoted to the enumeration of unrooted maps on an orientable surface of arbitrary genus g with a given number of edges n (see [12]). In the second part of the present work we use the approach described in [11,12] and the results obtained in the first part to enumerate unrooted 4-regular unicellular maps of a given genus g.

2. Statement of the problem

For the problem of enumeration it is convenient to use the combinatorial definition of a unicellular map. Namely, any one-face map M having n edges and k vertices could be specified by a triple (H, α, σ), where H describes the set of semi-edges of M, $|H| = 2n$. Cycles of the fix-point-free involution α define the edges of M, σ is a collection of k cycles ω_i describing dart orderings around the vertices of M, and $\gamma = \alpha\sigma$ is a cyclic permutation that encodes the traversal order of darts for the only face. For example, Fig. 1 shows a one-face map of genus 2 with $n = 6$ edges and $k = 3$ vertices. This map is described by the permutations

$$\alpha = (1, 10)(2, 12)(3, 5)(4, 7)(6, 8)(9, 11), \qquad \sigma = (1, 12, 10, 9)(2, 5, 8, 11)(3, 7, 6, 4),$$
$$\gamma = \alpha \cdot \sigma = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12).$$

The famous Euler's Formula [18, p. 268] for maps

$$k - n + f = 2 - 2g$$

implies that for a one-face ($f = 1$) map the number n of edges and the number k of vertices uniquely determine the genus g of the surface into which it is embedded:

$$n - k = 2g - 1. \tag{1}$$

It makes it possible to determine for given α and σ the genus g of the corresponding one-face map M and, conversely, for given g and n compute the number k of its vertices.

From Handshaking Lemma [18, p. 35] that could be written in the above terms as

$$2n = \sum_i |\omega_i| \tag{2}$$

it follows that for a 4-regular map the number of vertices is $k = n/2$. Taking into account Formula (1), we obtain that the number n of edges and the number k of vertices are given by

$$n = 4g - 2, \qquad k = 2g - 1. \tag{3}$$