European Journal of Combinatorics 62 (2017) 183–196

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Set families with a forbidden pattern

Ilan Karpas [a], Eoin Long [b]

[a] The Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

[b] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel

ARTICLE INFO

Article history:
Received 16 January 2016
Accepted 15 November 2016
Available online 16 January 2017

ABSTRACT

A *balanced pattern* of order $2d$ is an element $P \in \{+,-\}^{2d}$, where both signs appear d times. Two sets $A, B \subset [n]$ form a P-pattern, which we denote by $\mathrm{pat}(A,B) = P$, if $A \triangle B = \{j_1, \ldots, j_{2d}\}$ with $1 \leq j_1 < \cdots < j_{2d} \leq n$ and $\{i \in [2d] : P_i = +\} = \{i \in [2d] : j_i \in A \setminus B\}$. We say $\mathcal{A} \subset \mathcal{P}[n]$ is P-free if $\mathrm{pat}(A,B) \neq P$ for all $A, B \in \mathcal{A}$. We consider the following extremal question: how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if $\mathcal{A}$ is P-free?

We prove a number of results on the sizes of such families. In particular, we show that for some fixed $c > 0$, if P is a d-balanced pattern with $d < c \log\log n$ then $|\mathcal{A}| = o(2^n)$. We then give stronger bounds in the cases when (i) P consists of $d+$ signs, followed by $d-$ signs and (ii) P consists of alternating signs. In both cases, if $d = o(\sqrt{n})$ then $|\mathcal{A}| = o(2^n)$. In the case of (i), this is tight.

1. Introduction

A central goal in extremal set theory is to understand how large a set family can be subject to some restriction on the intersections of its elements. Given $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, we say that a set family $\mathcal{A}$ is $\mathcal{L}$-intersecting if $|A \cap B| \in \mathcal{L}$ for all distinct $A, B \in \mathcal{A}$. Taking $\mathcal{L}_t = \{s \in \mathbb{N} : s \geq t\}$, a fundamental theorem of Erdős, Ko and Rado [6] shows that $\mathcal{L}_t$-intersecting families $\mathcal{A} \subset \binom{[n]}{k}$ satisfy $|\mathcal{A}| \leq \binom{n-t}{k-t}$, provided $n \geq n_0(k,t)$. Another important theorem due to Frankl and Füredi [8] shows that if $\mathcal{L}_{\ell,\ell'} := \{s < \ell \text{ or } s \geq k - \ell'\}$, then any $\mathcal{L}_{\ell,\ell'}$-intersecting family $\mathcal{A} \subset \binom{[n]}{k}$ satisfies

E-mail addresses: ilan.karpas@mail.huji.ac.il (I. Karpas), eoinlong@post.tau.ac.il (E. Long).

http://dx.doi.org/10.1016/j.ejc.2016.11.005
0195-6698/

$|\mathcal{A}| \le cn^{\max(\ell,\ell')}$, for some constant c depending on k, ℓ and ℓ'. See [2,3,7,10] for an overview of this extensive topic.

Here we are concerned with understanding the effect of restricting the *pattern* formed between elements of a set family. A *difference pattern* or *pattern* of order t is an element $P \in \{+,-\}^t$. Given such a pattern P, let $S_+(P) = \{i \in [t] : P_i = +\} \subset [t]$ and $s_+(P) = |S_+(P)|$. Define $S_-(P)$ and $s_-(P)$ analogously. Two sets $A, B \subset [n]$ form a *difference pattern* P if:

(i) $A \triangle B = \{j_1, \ldots, j_t\}$ with $j_1 < \cdots < j_t$, and
(ii) $\{i \in [t] : P_i = +\} = \{i \in [t] : j_i \in A \setminus B\}$.

We denote this by writing $\mathrm{pat}(A, B) = P$. A family of subsets $\mathcal{A} \subset \mathcal{P}[n]$ is *P-free* if $\mathrm{pat}(A, B) \ne P$ for all distinct $A, B \in \mathcal{A}$. In this paper we consider the following natural question: given a pattern P, how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if it is P-free?

First note the following simple observation. If $s_+(P) \ne s_-(P)$ then large P-free families exist. Indeed, if $|s_+(P) - s_-(P)| = m > 0$ then the following families are P-free:

$$\mathcal{B}_1 = \{A \subset [n] : |A| \in [0, m-1] \bmod 2m\}; \qquad \mathcal{B}_2 = \{A \subset [n] : |A| \in [m, 2m-1] \bmod 2m\}.$$

Clearly either $|\mathcal{B}_1| \ge 2^{n-1}$ or $|\mathcal{B}_2| \ge 2^{n-1}$. We will therefore focus on the case when $s_+(P) = s_-(P) = d$. We say that such patterns are *d-balanced*. For a balanced pattern P it is only possible that $\mathrm{pat}(A, B) = P$ if $|A| = |B|$. Thus, our question on balanced patterns essentially reduces to a question for uniform families. Given $0 \le k \le n$, define

$$f(n, k, P) := \max\left\{|\mathcal{A}| : P\text{-free families } \mathcal{A} \subset \binom{[n]}{k}\right\}.$$

Let $f(n, k, d) = \max\{f(n, k, P) : P \text{ is } d\text{-balanced}\}$. We will also write $\delta(n, k, P)$ and $\delta(n, k, d)$ for the corresponding extremal densities, i.e. $\delta(n, k, P) := f(n, k, P)/\binom{n}{k}$, and $\delta(n, k, d) := f(n, k, d)/\binom{n}{k}$. Note also that if $\mathcal{A} \subset \binom{[n]}{k}$ is P-free then the family $\mathcal{A}^c = \{[n] \setminus A : A \in \mathcal{A}\} \subset \binom{[n]}{n-k}$ is also P-free. Therefore $f(n, k, P) = f(n, n-k, P)$ and it suffices to bound $f(n, k, P)$ for $k \le n/2$.

Our first aim is to prove a density result for d-balanced patterns of small order. That is, we will show that for fixed d, any sequence of integers $\{k_n\}_{n=1}^{\infty}$ tending to infinity with n with $k_n \le n/2$ satisfies $\lim_{n\to\infty} \delta(n, k_n, d) = 0$. The condition that k is not fixed and tends to infinity with n will be crucial. This is different from the case in the Frankl–Füredi Theorem, which tells us that we can take some fixed $k \ge 2d-1$, $\ell = k-d$ and $\ell' = d-1$, and if $\mathcal{A} \subset \binom{[n]}{k}$ with $|\mathcal{A}| = \omega(n^{k-d})$ then there are $A, B \in \mathcal{A}$ with $|A \triangle B| = 2d$, i.e. A and B form a P-pattern for *some* d-balanced pattern P. Indeed, take any fixed $k := k(d)$, and consider the family $\mathcal{A}_0 \subset \binom{[n]}{k}$ given by

$$\mathcal{A}_0 = \left\{A \subset [n] : \left|A \cap \left(\frac{(i-1)n}{k}, \frac{in}{k}\right]\right| = 1 \text{ for all } i \in [k]\right\}.$$

Then $|\mathcal{A}_0| \ge c_k n^k$ for some absolute constant $c_k > 0$, but it is easily seen that $\mathcal{A}_0$ does not contain the pattern $++--$. Therefore, there does not exist a density theorem for d-balanced patterns in subsets of $\binom{[n]}{k}$ with fixed k, as in the Frankl–Füredi theorem.

Our first result shows that such a density theorem does hold for k growing with n.

Theorem 1. *Given $d, k, n \in \mathbb{N}$ with $2k \le n$ and taking $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$ we have*

$$\delta(n, k, d) \le a_d k^{-c_d}.$$

By our discussion above for fixed k we see that Theorem 1 is in a sense a 'high-dimensional' result. Also note that Theorem 1 shows there is a constant $c > 0$ with the property that if P is a d-balanced pattern with $d \le c \log\log n$ and $\mathcal{A} \subset \mathcal{P}[n]$ which is P-free, then $|\mathcal{A}| = o(2^n)$.

Let $\mathrm{IP}(d)$ denote the d-balanced pattern consisting of d plus signs, followed by d minus signs. We refer to these as *interval patterns*. Given the obstruction of $\mathrm{IP}(2)$ above, it is natural to ask for bounds on $f(n, k, \mathrm{IP}(d))$.