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A size-sensitive inequality for cross-intersecting families

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ABSTRACT

Two families $\mathcal{A}$ and $\mathcal{B}$ of k-subsets of an n-set are called cross-intersecting if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. Strengthening the classical Erdős–Ko–Rado theorem, Pyber proved that $|\mathcal{A}||\mathcal{B}| \le \binom{n-1}{k-1}^2$ holds for $n \ge 2k$. In the present paper we sharpen this inequality. We prove that assuming $|\mathcal{B}| \ge \binom{n-1}{k-1} + \binom{n-i}{k-i+1}$ for some $3 \le i \le k+1$ the stronger inequality

$$|\mathcal{A}||\mathcal{B}| \le \left(\binom{n-1}{k-1} + \binom{n-i}{k-i+1}\right) \times \left(\binom{n-1}{k-1} - \binom{n-i}{k-1}\right)$$

holds. These inequalities are best possible. We also present a new short proof of Pyber's inequality and a short computation-free proof of an inequality due to Frankl and Tokushige (1992).

1. Introduction

Let $[n] = \{1, \dots, n\}$ and $\binom{[n]}{k}$ be the family of all k-subsets of $[n]$ for $n \ge k \ge 0$. For a family $\mathcal{F} \subset \binom{[n]}{k}$ let $\mathcal{F}^c$ be the family of complements, i.e., $\mathcal{F}^c = \{[n] - F : F \in \mathcal{F}\}$. Obviously, $\mathcal{F}^c \subset \binom{[n]}{n-k}$ holds.

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Two families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. To avoid trivialities we assume $n \geq 2k$. Analogously, $\mathcal{C}, \mathcal{D} \subset \binom{[n]}{l}$ are called *cross-union* if $C \cup D \neq [n]$ holds for all $C \in \mathcal{C}, D \in \mathcal{D}$. Here we assume $n \leq 2l$ in general. Note that $\mathcal{A}, \mathcal{B}$ are cross-intersecting iff $\mathcal{A}^c, \mathcal{B}^c$ are cross-union.

In order to state one of the most fundamental theorems in *extremal set theory*, let us say that $\mathcal{F} \subset \binom{[n]}{k}$ is *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

Theorem (*Erdős–Ko–Rado Theorem [2]*). *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $n \geq 2k > 0$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} \text{ holds.} \tag{1}$$

Hilton and Milner [7] proved in a stronger form that for $n > 2k$ the only way to achieve equality in (1) is to take all k-subsets containing some fixed element of $[n]$.

If $\mathcal{F}$ is intersecting then $\mathcal{A} = \mathcal{F}, \mathcal{B} = \mathcal{F}$ are cross-intersecting. Therefore the following result is a strengthening of (1).

Theorem (*Pyber's Inequality [14]*). *Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting and $n \geq 2k$. Then*

$$|\mathcal{A}|\,|\mathcal{B}| \leq \binom{n-1}{k-1}^2 \text{ holds.} \tag{2}$$

Let us mention that the notion of cross-intersection is not just a natural extension of the notion of intersecting for two families, but it is also a very useful tool for proving results for *one* family. As a matter of fact, it was already used in the paper of Hilton and Milner [7]. This explains the interest in two-family versions of intersection theorems (cf. e.g. [1,12,15]).

The object of this paper is two-fold. First we provide a very short proof of (2). Then we use the ideas of this proof and some counting based on the Kruskal–Katona Theorem [10,8] to obtain the following sharper, best possible bounds.

Example 1. Let i be an integer and define $\mathcal{B}_i = \{B \in \binom{[n]}{k} : 1 \in B\} \cup \{B \in \binom{[n]}{k} : 1 \notin B, [2, i] \subset B\}$, $\mathcal{A}_i = \{A \in \binom{[n]}{k} : 1 \in A, [2, i] \cap A \neq \emptyset\}$. Note that $\mathcal{A}_i, \mathcal{B}_i$ are cross intersecting with

$$|\mathcal{A}_i| = \binom{n-1}{k-1} - \binom{n-i}{k-1}, \qquad |\mathcal{B}_i| = \binom{n-1}{k-1} + \binom{n-i}{k-i+1}.$$

The inequalities (3) and (4) given below show that the pair $(\mathcal{A}_i, \mathcal{B}_i)$ is extremal in the corresponding range.

Theorem 1. *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting, $n > 2k > 0$ and suppose $|\mathcal{A}| \leq \binom{n-1}{k-1} \leq |\mathcal{B}|$ and $\bigcap_{B \in \mathcal{B}} B = \emptyset$. Then*

$$|\mathcal{A}|\,|\mathcal{B}| \leq \left(\binom{n-1}{k-1} + 1\right)\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1}\right) \text{ holds.} \tag{3}$$

Theorem 2. *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting, $n \geq 2k > 0$ and suppose that $|\mathcal{B}| \geq \binom{n-1}{k-1} + \binom{n-i}{k-i+1}$ holds for some $3 \leq i \leq k+1$. Then*

$$|\mathcal{A}|\,|\mathcal{B}| \leq \left(\binom{n-1}{k-1} + \binom{n-i}{k-i+1}\right)\left(\binom{n-1}{k-1} - \binom{n-i}{k-1}\right). \tag{4}$$