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Scribability problems for polytopes

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ABSTRACT

In this paper we study various scribability problems for polytopes. We begin with the classical k-scribability problem proposed by Steiner and generalized by Schulte, which asks about the existence of d-polytopes that cannot be realized with all k-faces tangent to a sphere. We answer this problem for stacked and cyclic polytopes for all values of d and k. We then continue with the weak scribability problem proposed by Grünbaum and Shephard, for which we complete the work of Schulte by presenting non weakly circumscribable 3-polytopes. Finally, we propose new (i, j)-scribability problems, in a strong and a weak version, which generalize the classical ones. They ask about the existence of d-polytopes that cannot be realized with all their i-faces "avoiding" the sphere and all their j-faces "cutting" the sphere. We provide such examples for all the cases where $j - i \leq d - 3$.

1. Introduction

The history of scribability problems goes back to at least 1832, when Steiner asked whether every 3-polytope is inscribable or circumscribable [43]. A polytope is *inscribable* if it can be realized with all its vertices on a sphere, and *circumscribable* if it can be realized with all its facets tangent to a sphere. Steiner's problem remained open for nearly 100 years, until Steinitz showed that inscribability and circumscribability are dual through polarity, and presented a technique to construct infinitely many non-circumscribable 3-polytopes [44]. A full characterization of inscribable 3-polytopes had to wait still more than 60 years, until Rivin gave one in terms of hyperbolic dihedral angles [36] (see also [25,34,35,37]). This was recently expanded by Danciger, Maloni and Schlenker [15], who obtained Rivin-style characterizations for 3-polytopes that are inscribable in a cylinder or a hyperboloid.

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A natural generalization in higher dimensions is to consider realizations of d-polytopes with all their k-faces tangent to a sphere. A polytope with such a realization is said to be *k-scribable*. This concept was studied by Schulte [40], who constructed examples of d-polytopes that are not k-scribable for all the cases except for $k = 1$ and $d = 3$ and the trivial cases of $d \leq 2$. In fact, every 3-polytope has a realization with all its edges tangent to a sphere. This follows from Koebe–Andreev–Thurston's remarkable Circle Packing Theorem [3,4,27,46] because edge-scribed 3-polytopes are strongly related to circle packings; see [51] for a nice exposition, and [12] for a discussion in higher dimensions. This was later generalized by Schramm [39], who showed that an edge-tangent realization exists even if the sphere is replaced by an arbitrary strictly convex body with smooth boundary.

Schulte [40] also proposed a weak version of k-scribability, following an idea of Grünbaum and Shephard [24]. A d-polytope is *weakly k-scribable* if it can be realized with the affine hulls of all its k-faces tangent to a sphere. Schulte was able to construct examples of d-polytopes that are not weakly k-scribable for all $k < d - 2$, and left open the cases $k = d - 2$ and $k = d - 1$.

Scribability problems expose the intricate interplay between combinatorial and geometric properties of convex polytopes. They arise naturally from several seemingly unrelated contexts. Inscribed polytopes are in correspondence with Delaunay subdivisions of point sets. This makes their combinatorial properties of great interest in computational geometry. They can also be interpreted as ideal polyhedra in the Klein model of the hyperbolic space. Moreover, several polytope constructions are based on the existence of k-scribed polytopes. For example, edge-scribed 4-polytopes are used by the *E-construction* to produce 2-simple 2-simplicial fat polytopes [18]. The *E_t-construction*, a generalization in higher dimension, exploits t-scribed polytopes [30].

However, our understanding on scribability properties is still quite limited. As Grünbaum and Shephard put it [24]: "it is surprising that many simple and tangible questions concerning them remain unanswered".

In this paper we study classical *k-scribability* problems, as well as a generalization, *(i, j)-scribability*, in both their strong and weak forms. The latter is the main object of interest of this paper; many of our results about k-scribability arise as consequences of our findings on (i, j)-scribability. We focus on the existence problems. Hence, for a family of polytopes, we either construct scribed realizations for each of them, or find explicit instances that are not scribable. As we have seen, scribability problems lie in the confluence of polyhedral combinatorics, sphere configurations, and hyperbolic geometry; and our proof techniques and constructions draw from all these areas.

1.1. k-scribability

Our investigation on the strong k-scribability problem focuses on two important families of polytopes: *stacked polytopes* and *cyclic (and neighborly) polytopes*.

By Barnette's Lower Bound Theorem [7,8], stacked polytopes have the minimum number of faces among all simplicial polytopes with the same number of vertices. The *triakis tetrahedron* is a stacked polytope among the first and smallest examples of non-inscribable polytopes found by Steinitz [44]. Recently, Gonska and Ziegler [22] completely characterized inscribable stacked polytopes. On the other hand, Eppstein, Kuperberg and Ziegler [18] showed that stacked 4-polytopes are essentially not edge-scribable.

In Section 5, we look at the other side of the story and prove the following result, which completely answers the k-scribability problems for stacked polytopes.

Theorem 1. *For any $d \geq 3$ and $0 \leq k \leq d - 3$, there are stacked d-polytopes that are not k-scribable. However, every stacked d-polytope is $(d - 1)$-scribable (i.e. circumscribable) and $(d - 2)$-scribable (i.e. ridge-scribable).*

The proof of Theorem 1 is divided into three parts: Proposition 5.1 for $0 \leq k \leq d-3$, Proposition 5.5 for $k = d - 2$ and Proposition 5.3 for $k = d - 1$. The construction for ridge-scribability extends in much more generality to connected sums of polytopes. The circumscribability statement is equivalent to saying that truncated polytopes are always inscribable. In Proposition 5.3, we show a stronger statement, that truncated polytopes are inscribable in any strictly convex surface. Hence, truncated polytopes are new examples of *universally inscribable* polytopes in the sense of [21].