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The planar cubic Cayley graphs of connectivity 2

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ABSTRACT

We classify the planar cubic Cayley graphs of connectivity 2, providing an explicit presentation and embedding for each of them. Combined with Georgakopoulos (2017) this yields a complete description of all planar cubic Cayley graphs.

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1. Introduction

This paper is the first in a series of related papers on planar Cayley graphs [9–11]. Combined with [10], it provides a complete description of the *cubic* planar Cayley graphs, i.e. those in which every vertex is adjacent with precisely three other vertices. This analysis provides surprising new examples, contradicting past conjectures, but also a good insight into planar Cayley graphs in general as explained [10]. Indeed, the ideas used in the cubic case are extended in [11] to arbitrary planar Cayley graphs. This provides in particular an effective enumeration of all planar Cayley graphs, settling a corresponding question of [6,7].

The aim of the current paper is a classification of the planar cubic Cayley graphs of connectivity 2, yielding an explicit presentation and embedding for each of those graphs. The *connectivity* of a graph G is the smallest cardinality of a set of vertices separating G. Cayley graphs of connectivity 1 are easy to describe. It is very common [7,8] to consider graphs of connectivity 2 separately from graphs of higher connectivity when studying planar Cayley graphs, and this is so for a good reason: by a classical theorem of Whitney [13, Theorem 11], planar graphs of connectivity at least 3 have an essentially unique embedding in the sphere S^2, and can thus be analysed taking advantage of this fact.

Our main result is

Theorem 1.1. *Let G be a planar cubic Cayley graph of connectivity 2. Then precisely one of the following is the case:*

(i) $G \cong Cay\langle a, b \mid b^2, (ab)^n\rangle, n \geq 2;$

(ii) $G \cong Cay\langle a, b \mid b^2, (aba^{-1}b^{-1})^n\rangle, n \geq 1;$

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Table 1
Classification of the cubic planar Cayley graphs with 2 generators and connectivity 2 (types (i) to (iii) of Theorem 1.1).

$G = Cay\langle a, b \mid b^2, \dots\rangle$, G is planar, and $\kappa(G) = 2$		
$a^n = 1$.	$a^n \neq 1$. Then G has a consistent embedding σ in which a preserves spin.	
$\Gamma \cong \langle a, b \mid b^2, a^4, (a^2b)^n\rangle$, $n \geq 2$. Thus G has a consistent embedding in which a reverses spin and b preserves spin.	$\Gamma \cong \langle a, b \mid b^2, (ab)^n\rangle$, $n \geq 2$ and b preserves spin in σ.	$\Gamma \cong \langle a, b \mid b^2, (aba^{-1}b^{-1})^n\rangle$, $n \geq 1$ and b reverses spin in σ.

(iii) $G \cong Cay\langle a, b \mid b^2, a^4, (a^2b)^n\rangle$, $n \geq 2$;

(iv) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (bcd)^m\rangle$, $m \geq 2$;

(v) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m\rangle$, $n, m \geq 2$;

(vi) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bd)^m\rangle$, $n, m \geq 2$;

(vii) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m\rangle$, $n, m \geq 2$;

(viii) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bcbd)^m\rangle$, $m \geq 1$;

(ix) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, cd\rangle$, $n \geq 1$ *(degenerate cases with redundant generators and G finite).*

Conversely, each of the above presentations, with parameters chosen in the specified domains, yields a planar cubic Cayley graph of connectivity 2.

Except for the above presentations we also construct embeddings of these graphs such that the corresponding group action on the graph carries facial walks to facial walks (Corollary 5.1); see also Section 2.3. Tables 1 and 2 summarise some information about these embeddings and some basic properties of the graphs, yielding a more structured presentation of the various possibilities of Theorem 1.1. The interested reader will also find further information, not included in these tables, throughout the course of the proofs. Moreover, in the last section of the paper we point out some interesting corollaries, which are extended in [10] to all cubic planar Cayley graphs.

This paper contributes to the complete classification of the cubic planar Cayley graphs of [10] not only by settling the special case of graphs of connectivity two, but also by providing building blocks for the construction of some of the 3-connected ones. For example, the graphs of type (vi) of Theorem 1.1 and their embeddings that we construct here are used in [10] to produce 3-connected Cayley graphs that, quite surprisingly, have no finite face boundaries.

In many cases we obtain embeddings of our graphs in the sphere using only the fact that their connectivity is 2, without assuming a priori that the graphs are planar. Thus we obtain the following result, which describes the non-planar cubic Cayley graphs of connectivity 2. A *hinge* of G is an edge $e = xy$ such that the removal of the pair of vertices x, y disconnects G.

Corollary 1.2. *Let G be a non-planar cubic Cayley graph of connectivity* 2. *Then one of the following is the case:*

(i) $G \cong Cay\langle a, b \mid b^2, a^n, \dots\rangle$, $n > 2$; *or*
(ii) $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^k, \dots\rangle$, $k \geq 3$, *G has no hinge, and every bc cycle contains a pair x, y of vertices separating the graph such that $x^{-1}y$ is an involution.*

2. Definitions

2.1. Cayley graphs and group presentations

We will follow the terminology of [5] for graph-theoretical terms and that of [2] for group-theoretical ones. Let us recall the definitions most relevant for this paper.

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