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Decomposing the complete r-graph

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ABSTRACT

Let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge set of the complete r-uniform hypergraph on n vertices. Graham and Pollak showed that $f_2(n) = n - 1$. An easy construction shows that $f_r(n) \leq (1 - o(1))\binom{n}{\lfloor r/2 \rfloor}$ and it has been unknown if this upper bound is asymptotically sharp. In this paper we show that $f_r(n) \leq (\frac{14}{15} + o(1))\binom{n}{r/2}$ for each even $r \geq 4$.

1. Introduction

The edge set of K_n, the complete graph on n vertices, can be partitioned into $n - 1$ complete bipartite subgraphs: this may be done in many ways, for example by taking $n - 1$ stars centred at different vertices. Graham and Pollak [7,8] proved that the number $n - 1$ cannot be decreased. Several other proofs of this result have been found, by Tverberg [12], Peck [9], and Vishwanathan [13,14].

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Generalising this to hypergraphs, for $n \geq r \geq 1$, let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge set of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices (i.e., the collection of all r-sets from an n-set). Thus the Graham–Pollak theorem asserts that $f_2(n) = n - 1$. For $r \geq 3$, an easy upper bound of $\binom{n-\lceil r/2\rceil}{\lfloor r/2\rfloor}$ may be obtained by generalising the star example above. Indeed, having ordered the vertices, consider the collection of r-sets whose 2nd, 4th, ..., $(2\lfloor r/2\rfloor)$th vertices are fixed. This forms a complete r-partite r-graph, and the collection of all $\binom{n-\lceil r/2\rceil}{\lfloor r/2\rfloor}$ such is a partition of $K_n^{(r)}$. (There are many other constructions achieving the exact same value – see, for example, Alon's recursive construction in [3].)

Alon [3] showed that $f_3(n) = n - 2$. More generally, for each fixed $r \geq 1$, he showed that

$$\frac{2}{\binom{2\lfloor r/2\rfloor}{\lfloor r/2\rfloor}}(1+o(1))\binom{n}{\lfloor r/2\rfloor} \leq f_r(n) \leq (1-o(1))\binom{n}{\lfloor r/2\rfloor},$$

where the upper bound is from the construction above.

The best known lower bound for $f_r(n)$ was obtained by Cioabă, Kündgen and Verstraëte [5], who showed that $f_{2k}(n) \geq \frac{2\binom{n-1}{k}}{\binom{2k}{k}}$. For upper bounds for $f_r(n)$, the above construction is not sharp in general. Cioabă and Tait [6] showed that $f_6(8) = 9 < \binom{8-3}{3}$, and used this to give an improvement in a lower-order term, showing that $f_{2k}(n) \leq \binom{n-k}{k} - 2\left\lfloor \frac{n}{16}\right\rfloor \binom{\lfloor \frac{n}{2}\rfloor - k + 3}{k-3}$ for any $k \geq 3$. (We mention briefly that any improvement of $f_4(n)$ for any n will further improve the above upper bound. Indeed, one can check that $f_4(7) = 9 < \binom{7-2}{2}$, and this will imply that $f_r(n) \leq \binom{n-\lfloor r/2\rfloor}{\lfloor r/2\rfloor} - cn^{\lfloor r/2\rfloor - 1}$ for some positive constant c. But note that, again, this is only an improvement to a lower-order term.)

Despite these improvements, the asymptotic bounds of Alon have not been improved. Perhaps the most interesting question was whether the asymptotic upper bound is the correct estimate.

The aim of this paper is to show that the asymptotic upper bound is not correct for each even $r \geq 4$. In particular, we will show that

$$f_4(n) \leq \frac{14}{15}(1+o(1))\binom{n}{2},$$

and obtain the same improvement of $\frac{14}{15}$ for each even $r \geq 4$.

A key to our approach will be to consider a related question: what is the minimum number of products of complete bipartite graphs, that is, sets of the form $E(K_{a,b}) \times E(K_{c,d})$, needed to partition $E(K_n) \times E(K_n)$? There is an obvious guess, namely that we take the product of the complete bipartite graphs in the partitions of both K_ns. This gives a partition using $(n-1)^2$ products of complete bipartite graphs. But can we improve this? Writing $g(n)$ for the minimum value, it will turn out that, unlike for f_4, any improvement in the value of $g(n)$ for one n gives an asymptotic improve-