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Abelian sandpile model and Biggs–Merino polynomial for directed graphs

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ABSTRACT

We prove several results concerning a polynomial that arises from the sandpile model on directed graphs; these results are previously only known for undirected graphs. Implicit in the sandpile model is the choice of a sink vertex, and it was conjectured by Perrot and Pham that the polynomial $c_0 + c_1y + \dots c_ny^n$, where c_i is the number of recurrent classes of the sandpile model with level i, is independent of the choice of the sink. We prove their conjecture by expressing the polynomial as an invariant of the sinkless sandpile model. We then present a bijection between arborescences of directed graphs and reverse G-parking functions that preserves external activity by generalizing Cori–Le Borgne bijection for undirected graphs. As an application of this bijection, we extend Merino's Theorem by showing that for any Eulerian directed graph the polynomial $c_0 + c_1y + \dots c_ny^n$ is equal to the greedoid polynomial of the graph.

1. Introduction

To what extent do the known results for undirected graphs extend to directed graphs? Driven by this question, we consider a remarkable theorem of Merino López [29] that

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expresses a one variable specialization of the Tutte polynomial of an undirected graph in terms of the abelian sandpile model on the graph. In this paper, we show that this theorem can be extended to all Eulerian directed graphs, and a weaker version of the theorem can be extended to all directed graphs.

The *abelian sandpile model* is a dynamical system on a finite directed graph that starts with a number of chips at each vertex of the graph. If a vertex has at least as many chips as its outgoing edges, then we are allowed to *fire* the vertex by sending one chip along each edge leaving the vertex to the neighbors of the vertex. This model was introduced by Dhar [20] as a model to study the concept of self-organized criticality introduced in [5]. Since then it has been studied in several different field of mathematics. In graph theory it was studied under the name of chip-firing game [13,37]; it appears in arithmetic geometry in the study of the Jacobian of algebraic curves [6,28]; and in algebraic graph theory it relates to the study of potential theory on graphs [7,8].

It is common to study the sandpile model by specifying a vertex as the sink vertex. In this model, chips that end up at the sink vertex are removed from the process. For a strongly connected directed graph, this guarantees that the sandpile model terminates (i.e., when none of the vertices have enough chips to be fired) in finite time. After fixing a sink, one can study a special type of chip configurations with the following property. A chip configuration is *recurrent* if, for any arbitrary chip configuration as the initial state, one can add a finite amount of chips to each vertex so that the recurrent configuration is the state of the sandpile model when the process terminates.

It was conjectured by Biggs [9] and proved by Merino López [29] that, for any undirected graph G and any choice of the sink vertex s,

$$c_0 + c_1 y + \ldots + c_n y^n = y^{|E(G)|}\mathcal{T}(G; 1, y) \qquad \text{(Merino's Theorem)},$$

where $\mathcal{T}(G; x, y)$ is the Tutte polynomial of the graph and c_i is the number of recurrent configurations with $i - \deg(s)$ chips. (We remark that the extra factor $y^{|E(G)|}$ does not appear in the right side of [29, Theorem 3.6] as their left side differs from ours by the same factor.) As the Tutte polynomial is defined without any involvement of the vertex s, this implies that $c_0 + c_1 y + \ldots + c_n y^n$ does not depend on the choice of s.

The sink independence of the polynomial $c_0 + c_1 y + \ldots + c_n y^n$ was then extended to all Eulerian directed graphs by Perrot and Pham [33], and they observed that the same statement does not hold for non-Eulerian directed graphs. However, they conjectured that a variant of this polynomial has the sink independence property for all directed graphs.

Perrot and Pham defined an equivalence relation on the recurrent configurations (Definition 3.3), and they defined the total number of chips of an equivalence class to be the maximum of the total number of chips of configurations contained in the class. They conjectured that the sink independence property is true for the polynomial,

$$\mathcal{B}(G, s; y) := c'_0 + c'_1 y + \ldots + c'_n y^n,$$