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A categorification of the chromatic symmetric function

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ABSTRACT

The Stanley chromatic symmetric function X_G of a graph G is a symmetric function generalization of the chromatic polynomial and has interesting combinatorial properties. We apply the ideas from Khovanov homology to construct a homology theory of graded $\mathfrak{S}_n$-modules, whose graded Frobenius series $\mathrm{Frob}_G(q,t)$ specializes to the chromatic symmetric function at $q = t = 1$. This homology theory can be thought of as a categorification of the chromatic symmetric function, and it satisfies homological analogues of several familiar properties of X_G. In particular, the decomposition formula for X_G discovered recently by Orellana, Scott, and independently by Guay-Paquet, is lifted to a long exact sequence in homology.

1. Introduction

Let G be a graph with a vertex set $V(G) = \{v_1, \dots, v_n\}$ and an edge set $E(G)$. A *proper coloring* of G is a function $\kappa : V(G) \to \mathbb{N}$ such that $\kappa(v_i) \neq \kappa(v_j)$ if an edge is

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incident to both v_i and v_j. The *chromatic polynomial* $\chi_G(x)$ *of* G is a polynomial such that for every $k \in \mathbb{N}$, $\chi_G(k)$ is the number of proper colorings of G with k colors.

Definition 1.1. The *chromatic symmetric function of* G is defined to be

$$X_G(\mathbf{x}) = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}, \tag{1}$$

where the sum is over all proper colorings $\kappa : V(G) \to \mathbb{N}$ of G.

The polynomial X_G is a generalization of the chromatic polynomial χ_G in the sense that $X_G(1^k) = \chi_G(k)$ [11, Proposition 2.2], for $k \in \mathbb{N}$.

For each $i \in \mathbb{N}$, $\kappa^{-1}(i)$ is an independent set in G, so any permutation of $\mathbb{N}$ that fixes all but finitely many elements gives another proper coloring of G, and therefore X_G is a symmetric function, homogeneous of degree n. Following the standard notation used by Macdonald, let p_λ and s_λ respectively denote the power sum symmetric functions and the Schur symmetric functions. Given a subset of edges $F \subseteq E(G)$, its *partition type* $\lambda(F)$ is the partition associated to the sizes of the connected components of the subgraph of G induced by the edge set F. The following formula of Stanley [11, Theorem 2.5] can be proved by an inclusion–exclusion argument.

$$X_G = \sum_{F \subseteq E(G)} (-1)^{|F|} p_{\lambda(F)}. \tag{2}$$

This formula forms the basis of our categorification process.

Categorification can be thought as a way of replacing an n-category by an $(n+1)$-category; for example, lifting the Euler characteristic of a topological space to its homology. One of the most successful recent examples of categorification is the link homology [5], a new kind of link invariant that lifts the properties of the Jones polynomial and carries a rich algebraic structure. In this theory, every link is assigned bigraded homology groups whose Euler characteristic is the Jones polynomial, and, additionally, link cobordisms are assigned homomorphisms of homology groups. This categorification has been successfully used in determining topological properties of knots, and was used by Rasmussen [9] to give a purely combinatorial proof of the Milnor conjecture, also known as Kronheimer–Mrowka theorem.

Chromatic graph homology, developed by Helme-Guizon and Rong [4], is one of several categorifications of polynomial graph invariants. The construction follows that of Khovanov; a bigraded homology theory is associated to a graph and a commutative graded algebra in such a way that its graded Euler characteristic is the value of the chromatic polynomial at the q-dimension of the algebra. There are other categorifications of the chromatic polynomial, and all of them possess a long exact sequence of homology that lifts the deletion–contraction formula for the chromatic polynomial, in the same way that Khovanov homology lifts the skein relations for the Jones polynomial.