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Large $\{0, 1, \ldots, t\}$-cliques in dual polar graphs

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ABSTRACT

We investigate $\{0, 1, \ldots, t\}$-cliques of generators on dual polar graphs of finite classical polar spaces of rank d. These cliques are also known as Erdős–Ko–Rado sets in polar spaces of generators with pairwise intersections in at most codimension t. Our main result is that we classify all such cliques of maximum size for $t \leq \sqrt{8d/5} - 2$ if $q \geq 3$, and $t \leq \sqrt{8d/9} - 2$ if $q = 2$. We have the following byproducts.

(a) For $q \geq 3$ we provide estimates of Hoffman's bound on these $\{0, 1, \ldots, t\}$-cliques for all t.
(b) For $q \geq 3$ we determine the largest, second largest, and smallest eigenvalue of the graphs which have the generators of a polar space as vertices and where two generators are adjacent if and only if they meet in codimension at least $t + 1$. Furthermore, we provide nice explicit formulas for all eigenvalues of these graphs.
(c) We provide upper bounds on the size of the second largest maximal $\{0, 1, \ldots, t\}$-cliques for some t.

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1. Introduction

A *clique* of a graph is a set of pairwise adjacent vertices of a graph. Determining the maximum size of a clique, the so-called clique number, is a classical problem in graph theory. For some graphs, cliques are traditionally called Erdős–Ko–Rado (*EKR*) sets. EKR sets were introduced by Erdős, Ko, and Rado [8] in 1961 as a family Y of k-element subsets of $\{1, \ldots, n\}$ where the elements of Y pairwise intersect in at least t elements. Erdős, Ko, and Rado classified all such Y of maximum size for $t = 1$.

Theorem 1 *(Theorem of Erdős, Ko, and Rado). Let $n \geq 2k$. Let Y be an set of k-element subsets of $\{1, \ldots, n\}$ such that all elements of Y pairwise intersect in at least one element. Then*

$$|Y| \leq \binom{n-1}{k-1}$$

with equality for $n > 2k$ if and only if Y is the set of all k-sets containing a fixed element.

For general t and large n, the theorem looks as follows.

Theorem 2. *Let t, k, n be positive integers with $1 \leq t \leq k$ and $n \geq (t+1)(k-t+1)$. Let Y be a set of k-element subsets of $\{1, \ldots, n\}$ such that $|K \cap K'| \geq t$ for all $K, K' \in Y$. Then*

$$|Y| \leq \binom{n-t}{k-t}.$$

These tight upper bounds for all t on EKR sets of sets were given by Wilson in 1984 [24]. The classification of all examples of maximum size was completed by Ahlswede and Khachatrian in 1997 [1]. Many generalizations of the EKR problem exist for general t. For example for vector spaces [9,13,21] and permutation groups [16].

With one exception, no attempts were made until now to investigate EKR sets of finite classical polar spaces in the general case. This exception is the investigation of $\{0, 1, 2\}$-cliques of dual polar graphs by Brouwer and Hemmeter [4], where they classified all $\{0, 1, 2\}$-cliques on dual polar graphs in the non-Hermitian cases. This problem was modified by De Boeck [5] to EKR sets, where he classified EKR sets Y of planes (not necessarily generators) for $d \in \{3, 4, 5\}$ and $|Y| \geq 3q^4 + 3q^3 + 2q^2 + q + 1$. Here q is the *order* of the polar space, that is the order of its underlying field, and d is the rank of the polar space. For the more restricted problem in sense of Theorem 1 Stanton proved upper bounds in [19]. The largest examples were mostly classified by Pepe, Storme, and Vanhove in [18]. For the remaining open case see [15,17].

Define a (d, t)-*EKR set* of generators of a polar space of rank d to be a set Y of generators of the polar space such that $y_1, y_2 \in Y$ implies $\operatorname{codim}(y_1 \cap y_2) := d - \dim(y_1 \cap y_2) \leq t$. In this notation Brouwer and Hemmeter investigated $(d, 2)$-EKR sets of finite classical