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A crystal embedding into Lusztig data of type A ☆

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ABSTRACT

Let $\mathbf{i}$ be a reduced expression of the longest element in the Weyl group of type A, which is adapted to a Dynkin quiver with a single sink. We present a simple description of the crystal embedding of Young tableaux of arbitrary shape into $\mathbf{i}$-Lusztig data, which also gives an algorithm for the transition matrix between Lusztig data associated to reduced expressions adapted to quivers with a single sink.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated to a symmetrizable Kac–Moody algebra $\mathfrak{g}$. The negative part of $U_q(\mathfrak{g})$ has a basis called a canonical basis [16] or lower global crystal basis [7], which has many fundamental properties. The canonical basis forms a colored oriented graph $B(\infty)$, called a crystal, with respect to Kashiwara operators. The crystal $B(\infty)$ plays an important role in the study of combinatorial aspects of $U_q(\mathfrak{g})$-modules together with its subgraph $B(\lambda)$ associated to any integrable highest weight module $V(\lambda)$ with highest weight λ.

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Suppose that $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra with the index set I of simple roots. Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a sequence of indices in I corresponding to a reduced expression of the longest element in the Weyl group of $\mathfrak{g}$. A PBW basis associated to $\mathbf{i}$ is a basis of the negative part of $U_q(\mathfrak{g})$ [17], which is parametrized by the set $\mathcal{B}_{\mathbf{i}}$ of N-tuple of non-negative integers. One can identify $B(\infty)$ with $\mathcal{B}_{\mathbf{i}}$ since the associated PBW basis coincides with the canonical basis at $q = 0$ [21]. We call an element in $\mathcal{B}_{\mathbf{i}}$ an $\mathbf{i}$-Lusztig datum or Lusztig parametrization associated to $\mathbf{i}$.

Consider the map

$$\psi^{\mathbf{i}}_{\lambda} : \ B(\lambda) \otimes T_{-\lambda} \hookrightarrow \mathcal{B}_{\mathbf{i}} \ , \tag{1.1}$$

given by the $\mathbf{i}$-Lusztig datum of $b \in B(\lambda)$ under the embedding of $B(\lambda) \otimes T_{-\lambda}$ into $B(\infty)$, where $T_{-\lambda} = \{t_{-\lambda}\}$ is an abstract crystal with $\mathrm{wt}(t_{-\lambda}) = -\lambda$ and $\varphi_i(t_{-\lambda}) = -\infty$ for $i \in I$. In this paper, we give a simple combinatorial description of (1.1) when $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathbf{i}$ is a reduced expression adapted to a Dynkin quiver of type A_{n-1} with a single sink (Theorem 5.4). It is well-known that when $\mathbf{i}$ is adapted to a quiver with one direction, for example $\mathbf{i} = \mathbf{i}_0 = (1, 2, 1, 3, 2, 1, \ldots, n-1, \ldots, 1)$, the $\mathbf{i}_0$-Lusztig datum of a Young tableaux is simply given by counting the number of occurrences of each entry in each row. But the $\mathbf{i}$-Lusztig datum for arbitrary $\mathbf{i}$ is not easy to describe in general, and one may apply a sequence of Lusztig's transformations [17] or the formula for a transition map $R^{\mathbf{i}}_{\mathbf{i}_0} : \mathcal{B}_{\mathbf{i}_0} \to \mathcal{B}_{\mathbf{i}}$ by Berenstein–Fomin–Zelevinsky [2]. We remark that our algorithm for computing $\psi^{\mathbf{i}}_{\lambda}$ is completely different from the known methods, and hence provides an alternative description of $R^{\mathbf{i}}_{\mathbf{i}_0}$.

Let us explain the basic ideas in our description of $\psi^{\mathbf{i}}_{\lambda}$. Suppose that Ω is a quiver of type A_{n-1} with a single sink and $\mathbf{i}$ is adapted to Ω. Let $J \subset I$ be a maximal subset such that each connected component of the corresponding quiver $\Omega_J \subset \Omega$ has only one direction. Let $\mathfrak{g}_J$ be the maximal Levi subalgebra and $\mathfrak{u}_J$ the nilradical associated to J, respectively.

The first step is to prove a tensor product decomposition $\mathcal{B}_{\mathbf{i}} \cong B^J(\infty) \otimes B_J(\infty)$ as a crystal, where $B_J(\infty)$ is the crystal of the negative part of $U_q(\mathfrak{g}_J)$ and $B^J(\infty)$ is the crystal of the quantum nilpotent subalgebra $U_q(\mathfrak{u}_J)$. The isomorphism is just given by restricting the Lusztig datum to each part, and it is a special case of the bijection introduced in [1,21] using crystal reflections. Here we show that it is indeed a morphism of crystals by using Reineke's description of $B(\infty)$ in terms of representations of Ω [19]. We refer the reader to a recent work by Salibury–Schultze–Tingley [23] on $\mathbf{i}$-Lusztig data, which also implies the combinatorial description of Kashiwara operators on $B(\infty)$ used in this paper.

The next step is to construct an embedding of $B(\lambda) \otimes T_{-\lambda}$ into $B^J(\infty) \otimes B_J(\infty)$ using a crystal theoretic interpretation of Sagan and Stanley's skew RSK algorithm [20], which was observed in the author's previous work [12] (see also [13,14]), and using the embedding (1.1) in case of $\mathbf{i}$ adapted to a quiver with one direction. Hence we obtain an $\mathbf{i}$-Lusztig datum of a Young tableau for any $\mathbf{i}$ adapted to Ω. One may consider the