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Enumerating polytropes

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ABSTRACT

Polytropes are both ordinary and tropical polytopes. We show that tropical types of polytropes in $\mathbb{TP}^{n-1}$ are in bijection with cones of a certain Gröbner fan $\mathcal{GF}_n$ in $\mathbb{R}^{n^2-n}$ restricted to a small cone called the polytrope region. These in turn are indexed by compatible sets of bipartite and triangle binomials. Geometrically, on the polytrope region, $\mathcal{GF}_n$ is the refinement of two fans: the fan of linearity of the polytrope map appeared in [24], and the bipartite binomial fan. This gives two algorithms for enumerating tropical types of polytropes: one via general Gröbner fan software such as gfan, and another via checking compatibility of systems of bipartite and triangle binomials. We use these algorithms to compute types of full-dimensional polytropes for $n = 4$, and maximal polytropes for $n = 5$.

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1. Introduction

Consider the tropical min-plus algebra $(\mathbb{R}, \oplus, \odot)$, where $a \oplus b = \min(a, b)$, $a \odot b = a + b$. A set $S \subset \mathbb{R}^n$ is tropically convex if $x, y \in S$ implies $a \odot x \oplus b \odot y \in S$ for all $a, b \in \mathbb{R}$. Such sets are closed under tropical scalar multiplication: if $x \in S$, then $a \odot x \in S$. Thus,

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one identifies tropically convex sets in $\mathbb{R}^n$ with their images in the tropical affine space $\mathbb{TP}^{n-1} = \mathbb{R}^n \backslash (1, \ldots, 1)\mathbb{R}$. The tropical convex hull of finitely many points in $\mathbb{TP}^{n-1}$ is a tropical polytope. A tropical polytope is a polytrope if it is also an ordinary convex set in $\mathbb{TP}^{n-1}$ [16].

Polytropes are important in tropical geometry and combinatorics. They have appeared in a variety of contexts, from hyperplane arrangements [18], affine buildings [17], to tropical eigenspaces, tropical modules [3,5], and, semigroup of tropical matrices [12], to name a few. Their discovery and re-discovery in different contexts have granted them many names: they are the alcoved polytopes of type A of Lam and Postnikov [18], the bounded L-convex sets of Murota [19, §5], the image of Kleene stars in tropical linear algebra [3,5].

This work enumerates the tropical types of full-dimensional polytropes in $\mathbb{TP}^{n-1}$. The tropical type, first defined for general tropical polytopes by Develin and Sturmfels [10], contains combinatorial data on a polytrope, and refines the ordinary type defined by face posets. Since polytropes are special tropical simplices [16, Theorem 7], the number of tropical types of polytropes in $\mathbb{TP}^{n-1}$ is at most the number of regular polyhedral subdivisions of $\Delta_{n-1} \times \Delta_{n-1}$ by [10, Theorem 1]. However, this is a very weak bound, the actual number of types of polytropes is much smaller. Joswig and Kulas [16] pioneered the explicit computation of types of polytropes in $\mathbb{TP}^2$ and $\mathbb{TP}^3$ using the software polymake. They started from the smallest polytrope, which is a particular ordinary simplex [16], and recursively added more vertices in various tropical halfspaces. Their table of results and beautiful figures have been the source of inspiration for this work. Unfortunately, the published table in [16] has errors. For example, there are six, not five, distinct tropical types of full-dimensional polytropes in $\mathbb{TP}^3$ with maximal number of vertices, as discovered by Jiménez and de la Puente [15]. We recomputed Joswig and Kulas' result in Table 2.

In contrast to previous works [15,16], we think of polytropes as the set of feasible solutions to the all-pairs shortest path integer program, and enumerate them via the Gröbner approach [6,22,23]. In Section 2, we show that the tropical types of polytropes are in bijection with a subset of cones in the Gröbner fan $\mathcal{GF}_n$ of a certain toric ideal. While this is folklore to experts, the obstacle has been in characterizing these cones. Without such characterizations, brute force enumeration of all cones in $\mathcal{GF}_n$ is infeasible for $n = 5$ on a conventional desktop, even with symmetry taken into account.

We show that the full-dimensional polytrope cones in $\mathcal{GF}_n$ are contained in a small cone called the polytrope region. Our main result, Theorem 25, gives an indexing system for the polytrope cones in terms of sets of compatible bipartite binomials and triangles. Geometrically, we show that on the polytrope region, the fan $\mathcal{GF}_n$ equals the refinement of the fan of linearity of the polytrope map $\mathcal{P}_n$, and the bipartite binomial fan $\mathcal{BB}_n$. The latter fan is constructed as a refinement of finitely many fans, each of which is the coarsening of an arrangement linearly isomorphic to the braid arrangement. The open, full-dimensional cones are in bijection with polytropes in $\mathbb{TP}^{n-1}$ with maximal number of vertices. These results elucidate the structure of $\mathcal{GF}_n$ and give algorithms for polytrope enumeration. Specifically, one can either compute the Gröbner fan $\mathcal{GF}_n$ restricted to the