Journal of Combinatorial Theory, Series A 151 (2017) 51–60

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

# Counting lattice points in free sums of polytopes ☆

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A R T I C L E I N F O

*Article history:*
Received 9 January 2016
Available online 3 May 2017

*Keywords:*
Polytopes
Ehrhart theory
Free sums
Weighted Ehrhart polynomial

A B S T R A C T

We show how to compute the Ehrhart polynomial of the free sum of two lattice polytopes containing the origin $P$ and $Q$ in terms of the enumerative combinatorics of $P$ and $Q$. This generalizes work of Beck, Jayawant, McAllister, and Braun, and follows from the observation that the weighted $h^*$-polynomial is multiplicative with respect to the free sum. We deduce that given a lattice polytope $P$ containing the origin, the problem of computing the number of lattice points in all rational dilates of $P$ is equivalent to the problem of computing the number of lattice points in all integer dilates of all free sums of $P$ with itself.



Let $P$ and $Q$ be full-dimensional lattice polytopes containing the origin with respect to lattices $N_P \cong \mathbb{Z}^{\dim P}$ and $N_Q \cong \mathbb{Z}^{\dim Q}$ respectively. The **free sum** (also known as 'direct sum') $P \oplus Q$ is a full-dimensional lattice polytope containing the origin in the lattice $N_P \oplus N_Q$, defined by:

$$P \oplus Q = \operatorname{conv}((P \times 0_Q) \cup (0_P \times Q)) \subseteq (N_P \oplus N_Q)_{\mathbb{R}},$$

☆ This work was initiated and partially completed while the author was supported by a University of Sydney Postdoctoral Fellowship under the supervision of A/Prof Anthony Henderson. The author would like to thank Anthony Henderson for some useful discussions.

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http://dx.doi.org/10.1016/j.jcta.2017.04.004
0097-3165/

where $\operatorname{conv}(S)$ denotes the convex hull of a set $S$, $N_{\mathbb{R}} := N \otimes_{\mathbb{R}} \mathbb{R}$ for a lattice $N$, and $0_P, 0_Q$ denote the origin in $N_P, N_Q$ respectively.

The **Ehrhart polynomial** $f(P; m)$ of $P$ is a polynomial of degree $\dim P$ characterized by the property that $f(P; m) = \#(mP \cap N_P)$ for all $m \in \mathbb{Z}_{\geq 0}$ [6]. Our goal is to describe the Ehrhart polynomial of $P \oplus Q$ in terms of the enumerative combinatorics of $P$ and $Q$.

We first observe that $\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$ and $\{\#(\lambda Q \cap N_Q) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$ determine $\{\#(\lambda(P \oplus Q) \cap (N_P \oplus N_Q)) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$, and hence the set $\{\#(m(P \oplus Q) \cap (N_P \oplus N_Q)) \mid m \in \mathbb{Z}_{\geq 0}\}$, which is encoded by the Ehrhart polynomial of $P \oplus Q$ (see (9) for a partial converse). Indeed, this follows from the following observation: if $\partial_{\neq 0} P$ denotes the union of the facets of $P$ not containing the origin, then, by definition, for any $\lambda \in \mathbb{Q}_{\geq 0}$:

$$\#(\partial_{\neq 0}(\lambda P) \cap N_P) = \#(\lambda P \cap N_P) - \max_{0 \leq \lambda' < \lambda} \#(\lambda' P \cap N_P),$$

and

$$\partial_{\neq 0}(\lambda(P \oplus Q)) = \bigcup_{\substack{\lambda_P, \lambda_Q \geq 0 \\ \lambda_P + \lambda_Q = \lambda}} \partial_{\neq 0}(\lambda_P P) \times \partial_{\neq 0}(\lambda_Q Q), \tag{1}$$

where the right hand side is a disjoint union.

It will be useful to express the invariants above in terms of corresponding generating series. Firstly, the Ehrhart polynomial may be encoded as follows:

$$\sum_{m \geq 0} f(P; m) t^m = \frac{h^*(P; t)}{(1-t)^{\dim P + 1}}, \tag{2}$$

where $h^*(P; t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $\dim P$ with non-negative integer coefficients, called the $h^*$**-polynomial** of $P$ [10]. Secondly, let $M_P := \operatorname{Hom}(N_P, \mathbb{Z})$ be the dual lattice, and recall that the dual polyhedron $P^\vee$ is defined to be $P^\vee = \{u \in (M_P)_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P\}$. Let

$$r_P := \min\{r \in \mathbb{Z}_{>0} \mid rP^\vee \text{ is a lattice polyhedron}\}. \tag{3}$$

Note that since $(P \oplus Q)^\vee$ is the Cartesian product $P^\vee \times Q^\vee$, we have $r_{P \oplus Q} = \operatorname{lcm}(r_P, r_Q)$. Then one may associate a generating series encoding $\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$:

$$\sum_{\lambda \in \mathbb{Q}_{\geq 0}} \#(\partial_{\neq 0}(\lambda P) \cap N_P) t^\lambda = \frac{\widetilde{h}(P; t)}{(1-t)^{\dim P}}, \tag{4}$$

where $\widetilde{h}(P; t) \in \mathbb{Z}[t^{\frac{1}{r_P}}]$ is a polynomial of degree at most $\dim P$ with fractional exponents and non-negative integer coefficients, called the **weighted** $h^*$**-polynomial** of $P$.

**Example 1.** Let $N_P = \mathbb{Z}$ and let $P = [-2, 2]$, $Q = [-1, 3] = P + 1$. Then $r_P = 2$, $r_Q = 3$, and one may compute: