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The weighted complete intersection theorem

Yuval Filmus

Department of Computer Science, Technion — Israel Institute of Technology, Israel

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ABSTRACT

The seminal complete intersection theorem of Ahlswede and Khachatrian gives the maximum cardinality of a k-uniform t-intersecting family on n points, and describes all optimal families for $t \geq 2$. We extend this theorem to the weighted setting, in which we consider unconstrained families on n points with respect to the measure μ_p given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. Our theorem gives the maximum μ_p measure of a t-intersecting family on n points, and describes all optimal families for $t \geq 2$.

1. Introduction

The Erdős–Ko–Rado theorem [8], a basic result in extremal combinatorics, states that when $k \leq n/2$, a k-uniform intersecting family on n points contains at most $\binom{n-1}{k-1}$ sets; and furthermore, when $k < n/2$ the only families achieving these bounds are *stars*, consisting of all sets containing some fixed point.

The analog of the Erdős–Ko–Rado theorem for t-intersecting families, in which every two sets must have at least t points in common, was proved by Ahlswede and Khachatrian [3,5], who gave two different proofs (see also the monograph [1]). When $t \geq 2$, the optimal families are always of the form $\mathcal{F}_{t,r} = \{S : |S \cap [t+2r]| \geq t+r\}$, as had been conjectured by Frankl [9]. They also determined the maximum families under the

E-mail address: yuvalfi@cs.technion.ac.il.

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condition that the intersection of all sets in the family is empty [2], as well as the maximum non-uniform t-intersecting families [6] ("Katona's theorem"). They also proved an analogous theorem for the Hamming scheme [4].

Dinur and Safra [7] considered analogous questions in the weighted setting. They were interested in the maximum μ_p measure of a t-intersecting family on n points, where the μ_p measure is given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. When $p \leq 1/2$, they related this question to the setting of the original Ahlswede–Khachatrian theorem with parameters K, N satisfying $K/N \approx p$. A similar argument appears in work of Ahlswede–Khachatrian [6,4] in different guise. The μ_p setting has since been widely studied, and has been used by Friedgut [10] and by Keller and Lifshitz [12] to prove stability versions of the Ahlswede–Khachatrian theorem.

While not stated explicitly in either work, the methods of Dinur–Safra [7] and Ahlswede–Khachatrian [4] give a proof of an Ahlswede–Khachatrian theorem in the μ_p setting for all $p < 1/2$, without any constraint on the number of points. More explicitly, let $w(n,t,p)$ be the maximum μ_p-measure of a t-intersecting family on n points, and let $w_{\sup}(t,p) = \sup_n w(n,t,p)$. The techniques of Dinur–Safra and Ahlswede–Khachatrian show that when $\frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}$, $w_{\sup}(t,p) = \mu_p(\mathcal{F}_{t,r})$. This theorem is incomplete, for three different reasons: it describes $w_{\sup}(t,p)$ rather than $w(n,t,p)$, it only works for $p < 1/2$, and it doesn't describe the optimal families.

Katona [11] solved the case $p = 1/2$, which became known as "Katona's theorem". Ahlswede and Khachatrian gave a different proof [6], and their technique applies also to the case $p > 1/2$. We complete the picture by finding $w(n,t,p)$ for all n, t, p and determining all families achieving this bound when $t \geq 2$. We do this by rephrasing the two original proofs [3,5] of the Ahlswede–Khachatrian theorem in the μ_p setting. Curiously, whereas the classical Ahlswede–Khachatrian theorem can be proven using either of the techniques described in [3,5], our proof needs to use both.

2. Preliminaries

We will use $[n]$ for $\{1,\ldots,n\}$, and $\binom{[n]}{k}$ for all subsets of $[n]$ of size k. We also use the somewhat unorthodox notation $\binom{[n]}{\geq k}$ for all subsets of $[n]$ of size at least k. The set of all subsets of a set A will be denoted 2^A.

A *family on n points* is a collection of subsets of $[n]$. A family $\mathcal{F}$ is *t-intersecting* if any $A, B \in \mathcal{F}$ satisfy $|A \cap B| \geq t$. A family is *intersecting* if it is 1-intersecting.

For any $p \in (0,1)$ and any n, the product measure μ_p is a measure on the set of subsets of $[n]$ given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$.

A family $\mathcal{F}$ on n points is *monotone* if whenever $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$. Given a family $\mathcal{F}$, its *up-set* $\langle \mathcal{F} \rangle$ is the smallest monotone family containing $\mathcal{F}$, consisting of all supersets of sets in $\mathcal{F}$.

For $n \geq t \geq 1$ and $p \in (0,1)$, the parameter $w(n,t,p)$ is the maximum of $\mu_p(\mathcal{F})$ over all t-intersecting families on n points, and the parameter $w_{\sup}(t,p)$ is given by