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Permutation groups and derangements of odd prime order ☆

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ABSTRACT

Let G be a transitive permutation group of degree n. We say that G is $2'$-elusive if n is divisible by an odd prime, but G does not contain a derangement of odd prime order. In this paper we study the structure of quasiprimitive and biquasiprimitive $2'$-elusive permutation groups, extending earlier work of Giudici and Xu on elusive groups. As an application, we use our results to investigate automorphisms of finite arc-transitive graphs of prime valency.

1. Introduction

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group on a finite set Ω of size at least 2. An element $x \in G$ is a *derangement* if it acts fixed-point-freely on Ω. Equivalently, if H is a point stabiliser, then x is a derangement if and only if the conjugacy class of x fails to meet H. An easy application of the Orbit-Counting Lemma shows that G contains

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derangements. This classical theorem of Jordan has interesting applications in number theory and topology (see Serre's article [24], for example).

By a theorem of Fein, Kantor and Schacher [11], G contains a derangement of prime power order. This result turns out to have some important number-theoretic applications; for example, it implies that the relative Brauer group of any nontrivial extension of global fields is infinite (see [11, Corollary 4]). It is worth noting that the existence of a derangement of prime power order in [11] requires the Classification of Finite Simple Groups. In most cases, G contains a derangement of prime order, but there are some exceptions, such as the 3-transitive action of the smallest Mathieu group M_{11} on 12 points. The transitive permutation groups with this property are called *elusive* groups, and they have been the subject of many papers in recent years; see [6,12,13,15,14,26], for example.

A local notion of elusivity was introduced in [5]. Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite transitive permutation group and let r be a prime divisor of $|\Omega|$. We say that G is r-*elusive* if it does not contain a derangement of order r (so G is elusive if and only if it is r-elusive for every prime divisor r of $|\Omega|$). In [5], all the r-elusive primitive almost simple groups with socle an alternating or sporadic group are determined. This work has been extended in our recent book [4], which provides a detailed study of r-elusive classical groups. The r-elusive notion leads naturally to the definition of a $2'$-elusive permutation group, which are the main focus of this paper.

Definition. A finite transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is $2'$-*elusive* if $|\Omega|$ is divisible by an odd prime, but G does not contain a derangement of odd prime order.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group with point stabiliser H. Recall that G is *primitive* if H is a maximal subgroup of G, and note that every nontrivial normal subgroup of a primitive group is transitive. This observation suggests a natural generalisation of primitivity; we say that G is *quasiprimitive* if every nontrivial normal subgroup is transitive. Similarly, G is *biquasiprimitive* if every nontrivial normal subgroup has at most two orbits on Ω, and there is at least one nontrivial normal subgroup with two orbits.

Quasiprimitive and biquasiprimitive groups arise naturally in the study of finite vertex-transitive graphs. For example, if G is a vertex-transitive group of automorphisms of a graph Γ such that for each vertex v, the action of the vertex stabiliser G_v on the set of neighbours of v is quasiprimitive (that is, Γ is G-*locally-quasiprimitive*), then [21, Lemma 1.6] implies that every normal subgroup N of G with at least three orbits is semiregular (that is, $N_v = 1$ for every vertex v). In this situation, the quotient graph with respect to the orbits of such a normal subgroup inherits many of the symmetry properties of the original graph Γ. This explains why quasiprimitive and biquasiprimitive groups often arise as base cases in the analysis of various families of vertex-transitive graphs, see for example [9,22]. These important graph-theoretic applications motivated Praeger to establish detailed structure theorems for quasiprimitive