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Minkowski complexes and convex threshold dimension

Florian Frick [a], Raman Sanyal [b]

[a] *Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*
[b] *Institut für Mathematik, Goethe-Universität Frankfurt, Germany*

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ABSTRACT

For a collection of convex bodies $P_1, \dots, P_n \subset \mathbb{R}^d$ containing the origin, a *Minkowski complex* is given by those subsets whose Minkowski sum does not contain a fixed basepoint. Every simplicial complex can be realized as a Minkowski complex and for convex bodies on the real line, this recovers the class of threshold complexes. The purpose of this note is the study of the *convex threshold dimension* of a complex, that is, the smallest dimension in which it can be realized as a Minkowski complex. In particular, we show that the convex threshold dimension can be arbitrarily large. This is related to work of Chvátal and Hammer (1977) regarding forbidden subgraphs of threshold graphs. We also show that convexity is crucial this context.

A simplicial complex Δ on vertices $[n] := \{1, \dots, n\}$ is a **threshold complex** if there are real numbers $\lambda_1, \dots, \lambda_n, \mu \in \mathbb{R}$ with $0 \le \lambda_i \le \mu$ for all $i = 1, \dots, n$ such that for any $\sigma \subseteq [n]$

$$\sigma \in \Delta \qquad \text{if and only if} \qquad \sum_{i \in \sigma} \lambda_i < \mu.$$

E-mail addresses: ff238@cornell.edu (F. Frick), sanyal@math.uni-frankfurt.de (R. Sanyal).

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Threshold complexes (or hypergraphs) where proposed by Golumbic [4] as a higher-dimensional generalization of the threshold *graphs* of Chvátal and Hammer [2]; see also [9]. If we assume that $0 \le \lambda_1 \le \cdots \le \lambda_n \le \mu$, then for any $i \in \sigma \in \Delta$ and $j < i$, we have $(\sigma \setminus i) \cup j \in \Delta$. Hence, threshold complexes are **shifted** in the sense of Kalai [6] and topologically wedges of (not necessarily equidimensional) spheres. See [7] and [3] for more information regarding the combinatorics and topology of threshold and shifted complexes.

The purpose of this note is to investigate a generalization of threshold complexes inspired by convex geometry. For that, let $\mathcal{P} = (P_1, \ldots, P_n)$ be an ordered family of convex bodies in $\mathbb{R}^d$ each containing the origin and let $\mu \in \mathbb{R}^d$ be a point. The **Minkowski complex** associated to $\mathcal{P}$ and μ is the simplicial complex $\Delta(\mathcal{P}; \mu)$ given by the simplices $\sigma \subseteq [n]$ with

$$\sigma \in \Delta(\mathcal{P}; \mu) \qquad \text{if and only if} \qquad \mu \notin P_\sigma := \sum_{i \in \sigma} P_i.$$

Here, $\sum_{i\in\sigma} P_i = \{\sum_{i\in\sigma} p_i : p_i \in P_i\}$ is the **Minkowski sum** (or vector sum) and we set $P_\emptyset := \{0\}$. By setting $P_i := \{t \in \mathbb{R} : 0 \le t \le \lambda_i\}$, it follows that threshold complexes are Minkowski complexes. For the case that each $P_i \subset \mathbb{R}^d$ is an axis-parallel box, these simplicial complexes have been studied by Pakianathan and Winfree [8] under the name of *quota complexes*. We may also replace a convex body P_i by a suitable convex polytope and we will tacitly do this henceforth.

Our motivation for studying Minkowski complexes comes from *mixed Ehrhart theory*. For a set $S \subset \mathbb{R}^d$, let us define the **discrete volume** $E(S) := |S \cap \mathbb{Z}^d|$. The **discrete mixed volume** of lattice polytopes $P_1, \ldots, P_n \subset \mathbb{R}^d$ is defined as

$$\mathrm{CM}E(P_1, \ldots, P_n) = \sum_{I \subseteq [n]} (-1)^{n-|I|} E(P_I).$$

It was shown in [5] (see also [1]) that, like its continuous counterpart the *mixed volume*, the discrete mixed volume satisfies

$$0 \le \mathrm{CM}E(Q_1, \ldots, Q_n) \le \mathrm{CM}E(P_1, \ldots, P_n)$$

for lattice polytopes $Q_i \subseteq P_i$ for $i = 1, \ldots, n$. Since $\mathrm{CM}E$ is invariant under lattice translations, we may assume that $0 \in P_i$ for all i. This allows us to express the discrete mixed volume as follows:

Theorem 1. *Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a family of $n > 0$ lattice polytopes in $\mathbb{R}^d$ with $0 \in P_i$ for all i. Then*

$$(-1)^n \mathrm{CM}E(P_1, \ldots, P_n) = \sum_{\mu \in P_{[n]} \cap \mathbb{Z}^d} \tilde{\chi}(\Delta(\mathcal{P}; \mu)),$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic.