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Power series with coefficients from a finite set ☆

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ABSTRACT

We prove in this paper that a multivariate D-finite power series with coefficients from a finite set is rational. This generalizes a rationality theorem of van der Poorten and Shparlinski in 1996.

1. Introduction

In his thesis [16], Hadamard began the study of the relationship between the coefficients of a power series and the properties of the function it represents, especially its singularities and natural boundaries. Two special cases of the problem have been extensively studied: one is on power series with integer coefficients and the other is on power series with finitely many distinct coefficients.

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In the first case, Fatou [13] in 1906 proved a lemma on rational power series with integer coefficients, which is now known as Fatou's lemma [33, p. 275]. The next celebrated result is the Pólya–Carlson theorem, which asserts that a power series with integer coefficients and of radius of convergence 1 is either rational or has the unit circle as its natural boundary. This theorem was first conjectured in 1915 by Pólya [25] and later proved in 1921 by Carlson [7]. Several extensions of the Pólya–Carlson theorem have been presented in [26,24,14,31,22,35,2].

In the second case, Fatou [13] was also the first to investigate power series with coefficients from a finite set by showing that such power series are either rational or transcendental. The study was continued by Pólya [25] in 1916, Jentzsch [17] in 1917, Carlson [6] in 1918 and finally Szegő [36,37] in 1922 settled the question by proving the following beautiful theorem (see [27, Chap. 11] and [10, Chap. 10] for its proof and related results).

Theorem 1 *(Szegő, 1922). Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{C}$. If F is continuable beyond the unit circle then it is a rational function of the form $F = P(x)/(1 - x^m)$, where P is a polynomial and m a positive integer.*

Szegő's theorem was generalized in 1945 by Duffin and Schaeffer [11] by assuming a weaker condition that f is bounded in a sector of the unit circle. In 2008, P. Borwein et al. in [5] gave a shorter proof of Duffin and Schaeffer's theorem. By using Szegő's theorem, van der Poorten and Shparlinski proved the following result [38].

Theorem 2 *(van der Poorten and Shparlinski, 1996). Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{Q}$. If $f(n)$ satisfies a linear recurrence equation with polynomial coefficients, then F is rational.*

A univariate sequence $f : \mathbb{N} \to K$ is *P-recursive* if it satisfies a linear recurrence equation with polynomial coefficients in $K[n]$. A power series $F = \sum f(n)x^n$ is *D-finite* if it satisfies a linear differential equation with polynomial coefficients in $K[x]$. By [32, Theorem 1.5], a sequence $f(n)$ is P-recursive if and only if the power series $F := \sum f(n)x^n$ is D-finite. The notion of D-finite power series can be generalized to the multivariate case (see Definition 4). Our main result is the following multivariate generalization of Theorem 2.

Theorem 3. *Let K be a field of characteristic zero, and let Δ be a finite subset of K. Suppose that $f : \mathbb{N}^d \to \Delta$ with $d \geq 1$ is such that*

$$F(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d} \in K[[x_1, \ldots, x_d]]$$

is D-finite. Then F is rational.