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Asymptotics of bivariate analytic functions with algebraic singularities

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we use the multivariate analytic techniques of Pemantle and Wilson to derive asymptotic formulae for the coefficients of a broad class of multivariate generating functions with algebraic singularities. Then, we apply these results to a generating function encoding information about the stationary distributions of a graph coloring algorithm studied by Butler, Chung, Cummings, and Graham (2015). Historically, Flajolet and Odlyzko (1990) analyzed the coefficients of a class of univariate generating functions with algebraic singularities. These results have been extended to classes of multivariate generating functions by Gao and Richmond (1992) and Hwang (1996, 1998), in both cases by immediately reducing the multivariate case to the univariate case. Pemantle and Wilson (2013) outlined new multivariate analytic techniques and used them to analyze the coefficients of rational generating functions. These multivariate techniques are used here to analyze functions with algebraic singularities.

1. Introduction

For several decades, singularity analysis has been used to derive asymptotic formulae for coefficients of univariate generating functions. In 1990, for example, Flajolet

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and Odlyzko found asymptotics for a large class of univariate functions with algebraic singularities in [4]. Examining the coefficients of multivariate generating functions is notoriously more difficult and technical. Pemantle and Wilson developed techniques to tackle multivariate rational generating functions in [10] and previous work, where they rely on the multivariate Cauchy integral, identifying and analyzing critical regions in the domain of integration that contribute to the integral's asymptotics through Morse theory. In this paper, we will look at the coefficients of $H(x,y)^{-\beta}$, where H is an analytic function and $\beta \notin \mathbb{Z}_{\leq 0}$ is a real number. Under some assumptions about the zero set of H, we will find an asymptotic approximation for the coefficients $[x^r y^s]H(x,y)^{-\beta}$ as r and s approach infinity with $\frac{r}{s}$ in a nearly-fixed ratio, as described in Theorem 1.

Flajolet and Odlyzko's 1990 results relied on using the Cauchy integral formula and explicit contour manipulations. Later in the 1990s, Gao, Richmond, Bender, and Hwang extended these results to classes of bivariate functions by temporarily fixing a variable and applying univariate results, which required special restrictions on the bivariate functions. (See Section 2.2 below for more details.) In this paper, we instead rely on the multivariate techniques that Pemantle and Wilson developed, manipulating the multivariate Cauchy integral formula directly. More details of these techniques are in Section 2.1 below. By using a combination of the Pemantle and Wilson techniques and the contour manipulations of the original Flajolet and Odlyzko work, we avoid using Morse theory. The algebraic singularities lead to manipulations of the torus on a Riemann surface instead of multidimensional complex space. However, this does not change the main methods of the asymptotic analysis, except requiring careful tracking of the argument of some expressions.

In Section 3, we state our main result (Theorem 1), which we prove in subsequent sections. Then, in Section 7, we look at examples of our results, including an application of Theorem 1 to a generating function that encodes properties of the stationary distributions of random colorings on the complete graphs, as found in [2].

An extended abstract of this paper, [7], appeared in the proceedings of the 28th International Conference on Formal Power Series and Algebraic Combinatorics.

2. Historical background

In this section, we provide some information about previous results in singularity analysis on which this work relies.

2.1. Multivariate analytic combinatorics of rational functions

In [10], Pemantle and Wilson outline a program which greatly extends the results of previous work on multivariate generating function analysis. Although many of the technical details of the program are not needed to prove the results in this paper, Pemantle and Wilson's work still lays the foundation for our approach. In the simplest case, Pemantle and Wilson begin with a rational function, $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$, where G and