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New bounds for partial spreads of $\mathsf{H}(2d-1, q^2)$ and partial ovoids of the Ree–Tits octagon

Ferdinand Ihringer [a], Peter Sin [b], Qing Xiang [c]

[a] *Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, 9190401, Israel*
[b] *Department of Mathematics, University of Florida, Gainesville, FL 32611, USA*
[c] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*

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A B S T R A C T

Two results are obtained that give upper bounds on partial spreads and partial ovoids respectively.
The first result is that the size of a partial spread of the Hermitian polar space $\mathsf{H}(3, q^2)$ is at most $((2p^3 + p)/3)^t + 1$, where $q = p^t$, p is a prime. For fixed p this bound is in $o(q^3)$, which is asymptotically better than the previous best known bound of $(q^3 + q + 2)/2$. Similar bounds for partial spreads of $\mathsf{H}(2d - 1, q^2)$, d even, are given.
The second result is that the size of a partial ovoid of the Ree–Tits octagon $\mathsf{O}(2^t)$ is at most $26^t + 1$. This bound, in particular, shows that the Ree–Tits octagon $\mathsf{O}(2^t)$ does not have an ovoid.

E-mail addresses: Ferdinand.Ihringer@gmail.com (F. Ihringer), sin@ufl.edu (P. Sin), qxiang@udel.edu (Q. Xiang).

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1. Introduction

Determining clique numbers of graphs is a traditional topic of combinatorics. Partial spreads of polar spaces are cliques of the oppositeness graph defined on the generators of the polar space. While bounds for partial spreads and partial ovoids are a long studied topic since Thas [11] popularized the problem, many questions remain open. The purpose of this paper is to address two of these open questions.

Our first result gives an improved bound on the size of partial spreads of certain Hermitian polar spaces. An *Hermitian polar space* $\mathsf{H}(2d-1, q^2)$, for $q = p^t$ a prime power, is the incidence geometry arising from a non-degenerate Hermitian form f of $\mathbb{F}_{q^2}^{2d}$. Here the flats of $\mathsf{H}(2d-1, q^2)$ consist of all nonzero totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form f; incidence is the inclusion relation of the flats of $\mathsf{H}(2d-1, q^2)$. The maximal totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form f are called *generators* of $\mathsf{H}(2d-1, q^2)$. A *partial spread* of $\mathsf{H}(2d-1, q^2)$ is a set of pairwise disjoint generators of $\mathsf{H}(2d-1, q^2)$. A simple double counting argument shows that a partial spread of $\mathsf{H}(2d-1, q^2)$ has size at most $q^{2d-1} + 1$. When d is odd, a better upper bound of $q^d + 1$ is known [14], and partial spreads of that size exist [1], [8]. So we are interested in the case when d is even.

Theorem 1.1. *Let $q = p^t$ with p prime and $t \geq 1$. Let Y be a partial spread of $\mathsf{H}(2d-1, q^2)$, where d is even.*

(a) *If $d = 2$, then $|Y| \leq \left(\frac{2p^3+p}{3}\right)^t + 1$.*
(b) *If $d = 2$ and $p = 3$, then $|Y| \leq 19^t$.*
(c) *If $d > 2$, then $|Y| \leq \left(p^{2d-1} - p\frac{p^{2d-2}-1}{p+1}\right)^t + 1$.*

The previous best known bound is in $\Theta(q^{2d-1})$ [6,7], while even for $d = 2$ the largest known examples only have size $\Theta(q^2)$ [4]. Here we provide the first upper bound which is in $o(q^{2d-1})$ for fixed p. For $d = 2$ the previous best known bound is $(q^3 + q + 2)/2$ [6]. An easy calculation shows that this old bound is better than the bound in part (a) of Theorem 1.1 if $p = 2$ and $t \leq 2$ or if $t = 1$. But for fixed p (and let $q = p^t$), the bound in part (a) of Theorem 1.1 is in $o(q^3)$, which is asymptotically better than the bound of $(q^3 + q + 2)/2$. For $d > 2$ the new bound improves all previous bounds if $t > 1$.

Our second result is a bound on the size of a partial ovoid in the *Ree–Tits octagons*. A *generalized n-gon* of order (s, r) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathsf{I})$, where elements of $\mathcal{P}$ are called *points*, elements of $\mathcal{L}$ are called *lines*, and $\mathsf{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an *incidence relation* between the points and lines, which satisfies the following axioms [13]:

(a) Each line is incident with $s + 1$ points.
(b) Each point is incident with $r + 1$ lines.
(c) The *incidence graph* has diameter n and girth $2n$.